CR Submanifolds in Kaehler and Nearly Kaehler Manifolds

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ABSTRACT. A review of the study of CR submanifolds within Kaehler and nearly Kaehler manifolds, and the properties of such manifolds with respect submanifold theory in differential geometry. The study in such a fashion is relatively young, most being carried out within the past thirty years. We consider CR submanifolds as a generalization of complex and real submanifolds, with the tangent bundle decomposing into real and complex parts. We demonstrate that the CR structure has strong consequences, and is heavily dependant on the properties of the ambient manifold.

The integrability of the real and complex parts is examined in various spaces, and we consider the existence of CR submanifolds with product, warped product, and foliate structure. The relationship governing the curvatures of the ambient manifold, the CR submanifold and leaves of the complex and real distributions are all considered.

We consider the general cases of complex, almost complex, Kaehler and nearly Kaehler manifolds. Further detail is included for the specific manifolds of flat complex space, complex hyperbolic space, complex projective space and the 6-sphere.

As an example of the applications of CR structure we include some work on the index of paths, and some topological consequences.

Examples of CR submanifolds are generated for the 6-sphere, and the properties of these submanifolds are considered, including the minimality and the second fundamental form. We include details of possible further study, and suggestions for how techniques used might be fruitfully employed elsewhere.

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Contents

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Part I

Acknowledgements

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Part II

Introduction

Submanifold theory has a long and deep history, being born in the classical notion of surfaces in three dimensional space. The more general theory of submanifolds has been well studied, and with many applications. If the ambient manifold has an almost complex structure it is natural to ask how the submanifold interacts with the complex structure, giving rise to the idea of real and holomorphic submanifolds.

This thesis is concerned with CR submanifolds - a class of submanifolds which is a generalisation of holomorphic and real submanifolds, but less well studied than either. While the behaviour and existence of holomorphic and real submanifolds has been examined extensively, CR submanifolds are relatively neglected despite having some interesting additional features. We study the existence of CR submanifolds, the restrictions on CR submanifolds dependant on the ambient manifold, and the properties of given CR submanifolds. It is our intention to demonstrate that CR sumanifolds are of interest in their own right, not a mere curious extension. We hope to show that the structure of CR submanifolds is deeply connected to the properties of the ambient manifold.

We are concerned mainly with the specific case of CR submanifolds in Kaehler and nearly Kaehler manifolds. The Kaehler structure will be shown to have immediate consequences for CR submanifolds. For nearly Kaehler structures we shall mainly consider the 6-sphere, S^6, interesting due to being simple enough to be well understood, whilst being complex enough to admit interesting submanifolds. The published work on holomorphic manifolds in both Kaehler manifolds and S^6 is extensive (see for example Berndt, Bolton, and Woodward[6]), and much CR submanifold study has been inspired by this existing work.

The main body of the thesis consists of a review of published work - we consider existence theorems for the case of Hermitian and Kaehler ambient manifolds in general and then some specific cases of well studied manifolds such as projective spaces, hyperbolic spaces and flat spaces. As an example of the use of the CR structure we shall recount some recent results concerning index theorems for CR submanifolds.

A large part is spent considering examples of CR submanifolds of S^6, culminating in an examination of warped-product CR submanifolds. The general construction of such submanifolds (with geodesic complex leaves) is original work due to the author and Dr John Bolton, inspired by the restricted examples demonstrated by Sekigawa, Hashimo and Mashimoto in [15] and [23]. We finish with some discussion of the properties of such submanifolds, and ideas for possible further study.

Proofs have been omitted in the cases where they are obvious or fully detailed on the referenced texts. We have also omitted proofs where they are overly long, or the ideas present have been previously indicated, and the presentation of the proof in full would not be of interest to the general reader. For example the proof of topolgocial results from Morse theory would be of limited interest. Where proofs have been reproduced we have added notes and comments, or expanded steps from the referenced texts.

Part III

Basic Concepts and Submanifold Theory

Chapter 1

Assumed Material and Some Notation

We shall be studying CR submanifolds, and hence will use some general results from Riemannian geometry, submanifold theory and complex structures. Some elements from other disciplines will also be required on an ad hoc basis (e.g. some results on Morse theory, group theory, homotopy theory), and will be quoted where required. Where a theorem or concept is used in isolation the defintions and related therems will be quoted where appropriate.

Basic differential geometry will be assumed, such as the defintion of a differentiable manifold, vectors, tensors, metrics, connections and so forth. Any introductory text on differential geometry will contain details of these concepts.

General theorems and definitions relating to submanifold theory, complex structures, and some other general concepts will be covered in the following chapters, as they will be referred to often. Some points where proofs and identities are standard the proofs have been omitted.

References for theorems will generally be quoted where required, however for general theory the reader is referred to Wilmore's book on Riemannian geometry [26], which contains many of the basic defintions and theorems required. We also refer the reader to Do Carmo[11], and Spivak[24] as containing much general theory for Riemannian geometry and submanifold theory.

Chapter 2

Basic Defintions and Submanifold Theory

2.1 Basic Defintions and Notation

Let M be a differentiable manifold. We shall then assume that M is of class C^∞. For a point p in M, we shall use T_pM to denote the tangent space at this point. We shall use TM to denote the tangent bundle over M, and use $\Gamma(TM)$ to denote the set of smooth vector fields over M (i.e. we shall use $X \in \Gamma(M)$ to indicate that X is a vector field over M).

We define a **metric** as a smooth non-degenerate bilinear map $g : T_pM \times T_pM \to \mathbf{R}$ for each $p \in M$. We also use the notation:

$$g(X,Y) = < X, Y > .$$

A metric is a **Riemannian metric**, if it is positive definite (i.e. $< X, X >$ is positive, except in the case where X is identically 0). A manifold equipped with a Riemannian metric is called a **Riemannian manifold.**

A **submanifold** is a manifold M equipped with an injective map $\psi : M \to \tilde{M}$, where $\tilde{M}$ is some other differentiable manifold called the **ambient manifold**. We further require that $d\psi$ is injective - where $d\psi$ maps the vector spaces $T_pM \to T_{\psi(p)}\tilde{M}$. If ψ is a homeomorphism preserving the topology of M, then the map is described as an **imbedding.** If ψ is a bijection, ψ and the inverse ψ^{-1} both C^∞ then the map called a **diffeomorphism.**

We shall often need to consider vectors over the submanifold M as vectors in $(\tilde{M})$. We therefore implicitly identify the vector space T_pM with the relevant subspace of $T_p\tilde{M}$.

We define a **distribution** D, to be a map $M \to$ (subsets of TM), s.t. $D(p) \subset T_pM^n$, such that each $D(p)$ is a vector subspace of T_pM, $dim\ D(p)$ is independent of p, and that D is a smooth map.

We use $\Gamma(D)$ to indicate the set of vector fields taking values in $D(p)$ at each point p. An **integral submanifold** of a distribution is a submanifold W s.t.

$T_pW \subset D(p) \forall p \in W$. A **maximal integral submanifold** is a submanifold W s.t. $T_pW = D(p) \forall p \in W$, not contained in any other integral submanifold. We normally indicate a maximal integral submanifold by some point in the ambient manifold - if an integral submanifold exists through a point, then there is a unique integrable submanifold.

We define a distribution as **involutive** if the distribution is closed under the action of the Lie bracket. We shall make free use of **Frobenius' Theorem** which states that any involutive distribution is integrable. (The converse is obvious)

2.2 Inherited Metrics, Connection, Riemannian Curvature, and the Second Fundamental Form

We shall in general be considering some Riemannian manifold $\tilde{M}$ (the **Ambient Manifold**), with submanifold M. We note that given this metric there is the natural Levi-Civita connection on $\tilde{M}$, denoted $\tilde{\nabla}$, the unique torsion free connection which is compatible with the metric. Associated with such a connection we have the **Riemannian curvature tensor, R**, given by :

$$\tilde{R}(X,Y)Z = \tilde{\nabla}_X \tilde{\nabla}_Y Z - \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]} Z, \tag{2.1}$$

where X, Y, Z are vector fields on $\tilde{M}$.

We assume all normal properties of the curvature tensor, the associated symmetries (e.g. anti-symmetry under X, Y above). We may also associate a **Ricci Tensor**, and a **scalar curvature** by the contractions:

$$R_{ij} = R^k_{ikj}, \tag{2.2}$$

$$R = R^i_i, \tag{2.3}$$

respectively, where we use normal index notation for these tensors, and assume contraction over repeated indices.

Given a submanifold M we naturally inherit a connection ∇ from the ambient manifold $\tilde{M}$ as follows. Given vectors $U, V \in T_pM$ any $p \in M$, then $U, V \in T_p\tilde{M}$ in a natural way, and so there is an induced metric on M. Hence there is an induced Levi-Civita connection ∇ on M given by this metric. Further we may show that ∇ is in fact just the tangential component on $\tilde{\nabla}$ when applied to vector fields in $\Gamma(TM)$.

Hence we may write :

$$\tilde{\nabla}_U V = \nabla_U V + h(U,V), \tag{2.4}$$

for $U, V \in \Gamma(TM)$ over M and $h(U,V)$ perpendicular to TM. This defines the tensor field h called the **Second Fundamental Form** of M in $\tilde{M}$. Note we have implicitly extended $U, V \in \Gamma(M)$ into fields in $\Gamma(\tilde{M})$ on some region of

$\tilde{M}$, but this is acceptable as it is easily shown that ∇ and h are independent of the extension chosen. The second fundamental form contains a great deal of information about how the submanifold is embedded in the manifold - we shall see that by using the Gauss equation the second fundamnetal form completely determines how the curvature tensor of the submanifold is related to the of the ambient manifold.

We also note that there is an associated **normal bundle** to M, which at each point $p \in M$ consists of that part of $T_p\tilde{M}$ perpendicular to T_pM. We also have a natural induced metric on this bundle, and an induced **normal connection** $\nabla^{\perp}$.

For ξ in the normal bundle, X tangential to M, we write:

$$\tilde{\nabla}_X\xi = -A_\xi X + \nabla^{\perp}_X\xi, \tag{2.5}$$

where the first term is tangential to M, and the second term perpendicular. This defines the operator A, which is related to the second fundamental form as follows:

$$< A_\xi U, V > = < h(U,V), \xi >, \tag{2.6}$$

where U, V are tangential fields to M, and ξ perpendicular to M. In some cases A is referred to as the **shape operator** of the submanifold. It may be shown that either of these two definitions of A are equivalent. The identity above relating A and h is someteimes referred to as the **Weingarten Equation.**

Observe that the tensor h is symmetric:

$$\begin{aligned}
h(U,V) - h(V,U) &= (\tilde{\nabla}_U V - \nabla_U V) - (\tilde{\nabla}_V U - \nabla_V U) && (2.7)\\
&= (\tilde{\nabla}_U V - \tilde{\nabla}_V U) - (\nabla_U V - \nabla_V U) && (2.8)\\
&= [U,V] - [U,V] && (2.9)\\
&= 0. && (2.10)\\
& && (2.11)
\end{aligned}$$

(Observe that the Lie bracket $[U, V]$ is conatained wholly in TM as the tangent bundle is necessarily integrable. Hence the evaluatiion is the same on the submanifold or in the ambient manifold).

We deduce that h may be diagonalised - let the entries of the diagonalisation of h be $r_1, ..., r_n$ called the pronicpal curvatures of the submanifold. We define two associated curvatures:

The **Mean Curvature**

$$H = \frac{1}{n}(r_1 + ...r_n),$$

and the **Gaussian Curvature**

$$K = r_1 r_2 ... r_n,$$

(the product of the principal curvatures).

It is of interest to note that the Gaussian curvature, although defined in terms of some embedding is actually an intrinsic property of the submanifold. It may be demonstrated that K is invariant under isometries of a surface.

We shall be particularly interested in the condition $H = 0$, defining **minimal submanifolds**. Such manifolds are studied classically as minimizing certain energy conditions, and most easily recognised as the surfaces formed by soap films.

We also note that there is also a naturally induced Riemannian curvature tensor, R, exactly as expected. However we will make use of several structural equations, referred to as equations of **Gauss** and **Codazzi** respectively.

$$< R(U,V)W, Z >=< \tilde{R}(U,V)W, Z > + < h(U,Z), h(V,W) > \tag{2.12}$$

$$- < h(U,W), h(V,Z) >, \tag{2.13}$$

$$(\tilde{R}(U,V)W) = (\nabla_U)h(V,W) - (\nabla_V)h(U,W), \tag{2.14}$$

where U, V, W, Z are tangential fields to M, and we assume the connection acts on h by:

$$(\nabla_U)h(V,W) = \nabla_U^{\perp}(h(V,W)) - h(\nabla_U V, W) - h(V, \nabla_U W).$$

A submanifold M is a **totally geodesic submanifold** in $\tilde{M}$ if every geodesic in M is a geodesic in $\tilde{M}$. Totally geodesic submanifolds are of interest because of their general simplicity, and the simple forms that induced curvatures, connections and the second fundamental forms necessarily take. For example in $\mathbf{R}^n$ planes through the origin are totally geodesic submanifolds. Similarly in the n-sphere S^n the totally geodesic submanifolds are precisely great spheres.

Chapter 3

Complex Structures

Given a real manifold $\tilde{M}$, we will define an **almost complex structure**, usually written as J, as a linear map on the tangent bundle $T\tilde{M}$ such that $J^2 = -Identity$. Observe that for such a structure to exist, the manifold must be of even dimension.

Observe that an n-dimensional complex manifold may be viewed as a 2n dimensional real manifold, in which case multiplication by i induces a complex structure. Observe that for local coordinate x_i we may write tangent vectors in the form $X = A_i \frac{\partial}{\partial x_i} + iB_j \frac{\partial}{\partial x_j}$ at a point. Multiplation by i maps X to $iX = -B_i \frac{\partial}{\partial x_i} + A_j \frac{\partial}{\partial x_j}$, and it is clear that such a transformation satisfies the requirements for an almost complex structure.

Hence every complex manifold generates an almost complex structure, but the converse is not true. We introduce the **Nijenhuis tensor**, N, by:

$$N(X,Y) = [JX, JY] - [X,Y] - J[JX,Y] - J[X,JY].$$

For vector fields X, Y over the manifold. The Nijenhuis tensor is identically zero if and only if the almost complex structure has been derived from a complex structure. We use the alternative notation:

$$N(X,Y) = [J,J](X,Y),$$

and extend the notation for arbitrary linear maps, ϕ of the tangent bundle, by:

$$[\phi,\phi](X,Y) = [\phi X, \phi Y] + \phi^2[X,Y] - \phi[\phi X, Y] - \phi[X, \phi Y].$$

If a manifold M admits an almost complex structure we shall call it an **Almost Complex Manifold** (As distinct from a complex manifold).

Given a Riemannian manifold $\tilde{M}$ with a submanifold M we will find it useful to decompose J into tangential and normal parts with respect to the submanifold M:

$$JU = \phi U + \omega U, \tag{3.1}$$
$$J\xi = B\xi + C\xi, \tag{3.2}$$
$$\tag{3.3}$$

where U is a tangential vector field to M, ξ perpendicular to M. The first terms, in ϕ and B, are tangential to M and the second terms, in ω and C are perpendicular to M.

If an almost complex structure is compatible with the metric as:

$$< JX, JY >=< X, Y >,$$

then we call the metric **Hermitian**. An almost complex manifold with Hermitian metric will be called **Almost Hermitian**. (And naturally a complex manifold with Hermitian metric is called a **Hermitian manifold**). We note in passing that any almost complex manifold can be given a Hermitian metric, e.g. by:

$$< X, Y >= g(X, Y) + g(JX, JY),$$

where g is any metric on the manifold. Almost all common manifolds of interest ($\mathbf{C}^n$, complex projective spaces, complex hyperbolic spaces, S^6) are Hermitian with the normal metric.

We define the **Kaehler form** on an almost Hermitian manifold $\tilde{M}$ by:

$$\Phi(X, Y) =< X, JY >,$$

for X, Y vector fields on $\tilde{M}$. If Φ is closed (i.e. $d\Phi = 0$) then we call $\tilde{M}$ a Kaehler manifold. An equivalent defintion is a manifold over which $\tilde{\nabla}_X J = 0$ for abitrary vector field X. We note that the equivalence of these two definitions is dependant on the Hermitian structure.

It is also possible to define a Kaehler metric in terms of a Kaehler potential, but we shall not be using this property.

There are several interesting Kaehler manifolds such as $\mathbf{CP}^n$, $\mathbf{C}^n$, and complex hyperbolic spaces.

The Kaehler property is of great use in manipulating terms involving the complex structure in connection terms. For example, consider some vector fields X, Y, then:

$$\begin{aligned} \tilde{\nabla}_X(JY) &= (\tilde{\nabla}_X J)Y + J(\tilde{\nabla}_X Y) && (3.4) \\ &= J\tilde{\nabla}_X Y. && (3.5) \end{aligned}$$

So we see that in Kaehler manifolds the complex structure may be pushed through covariant differentiation.

3.1 Nearly Kaehler Manifolds and S^6

We shall wish to consider manifolds such as S^6 which are not Kaehler, but instead have the property that:

$$(\tilde{\nabla}_X J)X = 0.$$

We call manifolds with this property **nearly Kaehler**. Note that various manipulations will fail in the nearly Kaehler case, for example:

$$\begin{aligned}\tilde{\nabla}_X(JY) &= (\tilde{\nabla}_X J)Y + J(\tilde{\nabla}_X Y), && (3.6)\\ & && (3.7)\end{aligned}$$

cannot be simplified further. However:

$$\begin{aligned}\tilde{\nabla}_X(JX) &= (\tilde{\nabla}_X J)X + J(\tilde{\nabla}_X X) && (3.8)\\ &= J\tilde{\nabla}_X X. && (3.9)\\ & && (3.10)\end{aligned}$$

This will have important consequences for the attempts to carry theorems over from the Kaehler case to the nearly Kaehler case.

The most common nearly Kaehler manifold, (often the only one considered) is S^6 with the almost complex structure defined as follows.

It is well known that the idea of complex numbers may be extended to four dimensional **quaternions**, of the form:

$$q = w + xi + yj + zk,$$

with $w, x, y, z \in \mathbf{R}$, and i, j, k satisfying the relations:

$$i^2 = j^2 = k^2 = -1,$$
$$ij = k,\ jk = i,\ ki = j,$$
$$ji = -k,\ kj = -i,\ ik = -j.$$

And conjugation by:

$$\bar{q} = w - xi - yj - zk.$$

We may further define an eight dimensional analogue, the **Octonians** (or sometimes called **Cayley numbers**), as ordered pairs of quaternions, i.e.

$$x = (q_1, q_2),$$

where q_i are quaternions. Addition and subtraction is defined as might be expected:

$$(q_1, q_2) \pm (q_3, q_4) = (q_1 \pm q_3, q_2 \pm q_4)$$

and multiplication by:

$$(q_1, q_2)(q_3, q_4) = (q_1 q_3 - \bar{q}_4 q_2, q_4 q_1 + q_2 \bar{q}_3).$$

Conjugation is defined by:

$$\overline{(q_1, q_2)} = (\bar{q}_1, -q_2).$$

Naturally we have a metric by $|x|^2 = x\bar{x}$. We define the real part of (q_1, q_2) to be the real part of q_1, and by extension a purely imaginary octonian to be one with identically zero real part.

We now identify the normal euclidean space $\mathbf{R}^7$ with the purely imaginary octonians, and define a wedge product (or vector product, or cross product), by:

$$x_1 \wedge x_2 = Im(x_1 x_2),$$

where the multiplication is to be carried out as defined above in the field of octonians. We may also define an inner product by the rela part of the multiplication, and observe that this is a natural extension of the concepts as defined in $\mathbf{R}^3$ - where the normal wedge and inner products may be identified as corresponding to multiplication in the field of quaternions. We note in passing that no higher dimensional algebras with these properties may be constructed - see Harvey and Lawson[14] for more details and a fuller treatment.

Let S^6 be the unit sphere in $\mathbf{R}^7$ with the normal euclidean metric, the inner and wedge products as defined above. Then for some point $p \in S^6$, and a vector $X \in T_p S^6$, we define:

$$JX = p \wedge X,$$

where we have identified $T_p M$ with the vector space in $\mathbf{R}^7$.

After some calculation we can demonstrate that J so defined is indeed an almost complex structure as $J^2 = -identity$. Further the Nijenhuis tensor is non vanishing, and hence S^6 with the given structure is not derived from a complex manifold. The given metric is Hermitian, and the almost complex structure is nearly Kaehler.

We note in passing that any 6-dimensional orientable manifold embedded in $\mathbf{R}^7$ may inherit this almost complex structure via the gauss map.

A large part of this work is concerned with the submanifolds of S^6 - this manifold is of interest due to its nearly Kahler structure (as opposed to Kaehler for many other interesting spaces), and the spheres S^n do not admit almost complex structures, except in the case $n = 2, 6$. As shall be shown CR submanifolds are only of interest in dimensions 3 and higher, requiring ambient spaces of dimension 4. Hence S^6 has a unique structure and construction, and admits many interesting results which are not found in other spaces.

Chapter 4

Totally Real, Complex, and CR Submanifolds

Given a almost complex manifold $\tilde{M}$ we are interested in how a submanifold M interacts with the almost complex structure J.

Traditionally two types have been studied - complex submanifolds and totally real submanifolds. We label a submanifold M as **complex** or **holomorphic** if its tangent bundle is preserved by J, i.e. $J : TM \to TM$. There has been extensive work identifying these types of submanifolds and studying their properties, for example Bolton, Vrancken, Woodward[8].

A submanifold M is called **totally real** if $J(TM)$ is perpendicular to TM. i.e. $< JX, Y >= 0, \forall X, Y \in TM$. A totally real submanifold is sometimes labelled **lagrangian** or **anti-holomorphic** if it is of maximal dimension - i.e. half that of the ambient manifold. As with complex submanifolds there is considerable literature on the existence and properties of such submanifolds.

This work is concerned with a generalisation of these ideas to CR submanifolds, as defined by Bejancu in [1].

M, a submanifold of $\tilde{M}$, is a **CR submanifold** if there exist orthogonal distributions $D, D^{\perp}$ over M s.t.

$$TM = D \oplus D^{\perp},$$

where D is preserved by J, and $D^{\perp}$ is mapped to $(TM)^{\perp}$ in $T\tilde{M}$, and $D, D^{\perp}$ are of constant dimension. It is very quickly seen that real and complex submanifolds are special cases where D or $D^{\perp}$ respectively, is the trivial zero-distribution. We therefore call the case excluding these, where both D and $D^{\perp}$ are both non-trivial, **proper CR submanifolds**. We see that a CR submanifold is an extension of complex and real submanifolds in that part of the tangent space acts as a real submanifold, and part as a complex submanifold. Hence we will sometimes refer to D as the complex distributions, and $D^{\perp}$ as the real distribution. It should be noted that not all submanifolds are of CR type - we shall give firm examples later in S^6, but submanifolds may fail (for example) due to the real part being non-perpendicular to the complex part, or the complex or real distributions being of non-constant dimension.

We shall sometimes find it useful to denote the vector bundle perpendicular to both TM and $J(TM)$ by ν, so that

$$T\tilde{M} = D \oplus D^{\perp} \oplus JD^{\perp} \oplus \nu.$$

In the case where ν is the zero chapter, (equivalently $D^{\perp}$ is of maximal dimension permitted by D) then the submanifold is called **anti-holomorphic**, an extension of the previous defintion.

Having defined the basic concept of CR submanifold we also find it useful to define certain properties that such a submanifold may have.

We will call a CR submanifold M a **CR product** if it is locally about $\forall p \in M$ the Riemannian product of submanifolds M' and $M^{\perp}$ respectively leaves of D and $D^{\perp}$. By definition therefore the distributions D and $D^{\perp}$ must be integrable on a CR product.

We define CR submanifold as **mixed geodesic** if $h(U, X) = 0\ \forall U \in D, X \in D^{\perp}$.

We describe a CR submanifold M as **mixed foliate** if D is integrable and M is mixed geodesic.

4.1 Levi Structure

We shall find it useful to use an adapted definition of the **Levi Structure** of a CR submanifold M, as defined by Lei and Wolfson in [18]. In [18] only submanifolds with real dimension (equal to $\dim(D^{\perp})$) 1 are considered. If we define a local frame on a CR submanifold $\{U_i, JU_i\}$ s.t. $U_i \in \Gamma(D)$ (so that there is no degeneracy in the framing), and $X \in \Gamma(D^{\perp})$, then we define the **Levi Form** c_{ij} by:

$$[U_i, JU_j] = c_{ij}X + \ terms\ in\ D.$$

Clearly the exact form of c_{ij} is dependent on the framing chosen, however we define a 2-form τ ($\tau_{ij} - c_{ij} inindexnotationinthisframing), andtherankandnullityoftheLeviform \tau$ are well defined.

The **nullity** of the submanifold is defined as the complex dimension of the null space of the Levi form. M is called **Levi flat** if the Levi-form vanishes. By referring to Frobenius' theorem we see that this corresponds to the case in which the distribution D is integrable.

We extend the definition of Levi form by using a local framing $\{X_i\}$ for the distribution $D^{\perp}$ and associating a form c_{ij}^k by:

$$[U_i, JU_j] = c_{ij}^k X_k + \ terms\ in\ D.$$

(Assuming summation over k).

Again the case where the Levi-form vanishes corresponds to the case in which both D and $D^{\perp}$ are integrable.

4.2 Alternative Definitions

We note that similar structures called variously **CR structures, CR manifolds** may be defined with respect to complex manifolds - see for example Jacobowitz[17], and indeed there has been a longer history of study of such structures. Although in many cases roughly equivalent, there are important differences in exact definitions and exact properties, and often the defintion is restricted to the case where the real part of the manifold is of dimension 1, and sometimes other conditions are imposed on the tangent spaces. Such objects naturally occur as (2n-1) dimensional real submanifolds in n-dimensional complex spaces.

We will not give the definition here, and will only be considering those submanifolds as defined above following Bejancu et al. We should note that these structures have strong relations to the Cauchy-Riemann equations over a complex manifold, and for this reason in this context CR is taken to stand for Cauchy-Riemann. Using CR submanifolds as a generalisation of real and complex submanifolds there is a case for considering CR as standing for Complex-Real in this situation.

For completeness we shall also note the definition of **Quaternionic submanifolds**, where the single linear map J, is extended to a set of linear maps J, K, L, in analogue with the extension of complex numbers to quaternionic numbers. In the same way we may define **QR-Submanifolds** as submanifolds with quaternionic and real distributions.

Chapter 5

Products and Warped Products

For general theory of warped products and twisted products, we refer the reader to the paper by Ponge and Reckziegel[20].

The most general case of a **twisted product** may be defined as follows: Let (M_1, g_1) , (M_2, g_2) be Riemannian manifolds with their associated metrics, and define smooth, positive functions:

$$f_1 : M_1 \times M_2 \to \mathbf{R},$$

$$f_2 : M_1 \times M_2 \to \mathbf{R}.$$

We define π_1, π_2 as the projections onto M_1, M_2. The twisted product $M_1 \times M_2$ is then a manifold equipped with the Riemannian metric g given at a point $p = (p_1, p_2)$, by:

$$g(X, Y) = f_1(p)g_1(\pi_1^* X, \pi_2^* Y) + f_2(p)g_2(\pi_2^* X, \pi_2^* Y).$$

We shall be more concerned with the restricted case where f_1 is the identity, and f_2 depends only on $p_1 \in M_1$. We denote this as the **warped product** of M_1 and M_2, and denote it by $M_1 \times_{f_2} M_2$. The case where f_2 is the identity corresponds to a **Riemannian product** manifold. We shall often refer to f_2 as the warping function of M_2 over M_1.

We apply these concepts to CR submanifolds by considering the case where D and $D^\perp$ are both integrable distributions, with each leaf in $D^\perp$ isometric to every other leaf of $D^\perp$, and each leaf of D isometric to every other leaf of D. We call M a **CR warped product** if it is a warped product $M_1 \times_f M_2$ with M_1 a holomorphic submanifold, and M_2 totally real. We shall call M a **CR product** if f is the identity, i.e. M is a Riemannian product. CR products, and CR warped products are natural objects to investigate as their structure is defined in terms of the simpler complex and real parts, and hence there is some idea of how to construct them, and sufficient structure to enable analysis to be carried out.

We will find the following theorem of use:

5.0.1 Theorem

Let M be a warped product submanifold of some Riemannian manifold as above, $M = M_1 \times_f M_2$, then M_1 is totally geodesic in M.

Proof Take some geodesic $\gamma(t)$ in M_1. Take arbitrary points $\gamma(t_0)$, $\gamma(t_1)$, then, for sufficiently short geodesics, the distance between these points is minimised in M_1. If this is so we now need to show that the length is minimized in M. Consider some small variation in M, $\delta_1(t)$, $\delta_2(t)$, respectively in M_1 and M_2. Then:

$$\begin{aligned}
(L + \delta L(t_0, t_1))^2 &= \int_{t_0}^{t_1} < \dot{\gamma}(t) + \dot{\delta}(t), \dot{\gamma}(t) + \dot{\delta}(t) > dt \qquad (5.1)\\
&= \int_{t_0}^{t_1} < \dot{\gamma}(t) + \dot{\delta}_1(t), \dot{\gamma}(t) + \dot{\delta}_1(t) >_1 + \qquad (5.2)\\
& \quad f(\gamma(t) + \delta_1(t), \gamma(t) + \delta_1(t)) < \dot{\delta}_2(t), \dot{\delta}_2(t) >_2 dt. \qquad (5.3)
\end{aligned}$$

Now observe that as M is a submanifold of a Riemannian manifold it necessarily has a positive definite metric, and so f is a strictly positive function. We therefore see that :

$$(\delta L(t_0, t_1)) > L(t_0, t_1) = (\int_{t_0}^{t_1} < \dot{\gamma}(t), \dot{\gamma}(t) > dt)^{1/2}.$$

And as this is true for arbitrary points on the geodesic, we see that γ is a geodesic in M, and hence M_1 is totally geodesic in M.//

We note that the same is not true for M_2 due to the fact that we may vary in M_1 so as to move to lower values for f. Consider for example the punctured sphere S^2 $\{poles\} = \mathbf{R} \times_f S^1$, where the totally geodesic $\mathbf{R}$ form great circles, but the warped S^1 form small circles, not geodesic in S^2.

We note as a consequence that if f is identically equal to one, then we have:

5.0.2 Theorem

If M is a product CR submanifold then D and $D^\perp$ are both integrable, and the complex and real leaves are each totally geodesic in M.

The converse is also true: If M is a CR submanifold with integrable D, $D^\perp$, such that the real and complex leaves are each totally geodesic in M, and each the integral submanifolds in D, $D^\perp$ are isometric respectively, then M is a CR product submanifold.

Proof Take M^c an arbitrary leaf of the distribution D, and observe that the distribution $D^\perp$ defines totally real manifolds through each point, hence M does indeed have a twisted product structure. Consider some geodesic in M^c, then if we move to adjacent leaves of the distribution the length of the geodesic varies depending on the warping function $f : M^\perp \to \mathbf{R}$, where $M/perp$ is the leaf of $D^\perp$. If f is not a local minimum (with respect to both the chosen geodesic and leaf M^c), then we may vary the geodesic so as to reduce its length in M, contradicting the assumption that all leaves of $D^\perp$ are totally geodesic in M. Hence f is a local minimum for this choice of leaf and geodesic - but as this choice was arbitrary we see that f is a loacl minimum for all leaves of D, and for all geodesics in each leaf, hence for all of $M^\perp$, and hence a constant. By a similar argument the warping function on $f' : M^c \to \mathbf{R}$ is also constant, and so we deduce that the manifold M is indeed a product manifold.//

Chapter 6

G_2 structure

G_2 is of interest as one of the exceptional Lie groups, and is a subject of much interest in its own right, and as might be expected may be defined and investigated in various ways. We shall be mostly intereted in how G_2 relates to the almost complex structure of S^6, and the consequences for CR submanifolds in S^6.

We define G_2 as the subgroup of $GL(\mathbf{R}, \mathbf{7})$ which preserves the standard euclidean metric, and the almost complex structure on S^6, with determinant $+1$. We quickly see that it is the subgroup of $SO(7)$ which preserves the wedge product on $\mathbf{R}^7$. i.e.

$$A(X \wedge Y) = AX \wedge AY, \forall A \in G_2, \ X, Y vectors \ in \ \mathbf{R}^7.$$

G_2 is a Lie group and hence analysis is helped by examining the associated Lie algebra g_2. We may define g_2 either as the set of left invarient vector fields on G_2, or equivalently the tangent space to G_2 at the identity.

We observe first of all that the metric preserving condition implies that $g_2 \subset so(7)$, and it is well known that $so(7)$ consists of 7×7 skew symmetric matrices. We must impose a further condition derived from preserving the complex structure as follows:

Let $A(s)$ be some path in G_2, s.t. $A(0)$ is the identity. We shall let e_i, e_j to be a basis for the tangent space of $\mathbf{R}^7$ at the identity. By the defintion of G_2:

$$A(s)(e_i \wedge e_j) = A(s)e_i \wedge A(s)e_j.$$

Differentiate this with respect to s, and evaluate at zero, to obtain an element of the tangent at the identity:

$$\begin{aligned} A'(0)(e_i \wedge e_j) &= A'(0)e_i \wedge A(0)e_j + A(0)e_i \wedge A'(0)e_j \\ A'(0)(e_i \wedge e_j) &= A'(0)e_i \wedge e_j + e_i \wedge A'(0)e_j. \end{aligned}$$

For any element of the tanagent space we may find some associated path in G_2 (by definition of the tangent space) and so every element $X \in g_2$ obeys this relation:

$$X(e_i \wedge e_j) = Xe_i \wedge e_j + e_i \wedge Xe_j.$$

In fact if $X \in so(7)$ this is a sufficient condition for $X \in g_2$. Thus

$$\begin{aligned} g_2 &= \{X \in gl(\mathbf{R}, 7) | X \in so(7), X(e_i \wedge e_j) = Xe_i \wedge e_j + e_i \wedge Xe_j \forall i, j = 1, ..., 7\} \\ &= \{X \in gl(\mathbf{R}, 7) | X^T = -X, X(e_i \wedge e_j) = Xe_i \wedge e_j + e_i \wedge Xe_j \forall i, j = 1, ..., 7\}. \end{aligned}$$

This is sufficient information to construct the general form of g_2. We start by setting the following relations:

$$\begin{aligned} Xe_1 &= a_1e_2 + a_2e_3 + a_3e_4 + a_4e_5 + a_5e_6 + a_6e_7, \\ Xe_2 &= -a_1e_1 + b_1e_3 + b_2e_4 + b_3e_5 + b_4e_6 + b_5e_7. \end{aligned}$$

Where the choice of $-a_1e_1$ as a factor in Xe_2 has been forced by the anti-symmetry requirement. We are free to choose these factors as the wedge product relations involve three basis elements. Consider therefore the relationship $e_1 \wedge e_2 = e_3$. If $X \in g_2$ we must fulfil:

$$Xe_3 = Xe_1 \wedge e_2 + e_1 \wedge Xe_2.$$

We can completely write out the right hand side:

$$\begin{aligned} Xe_3 = &(-a_2 + 0)e_1 + (0 - b_1)e_2 + 0e_3 + \\ &(a_5 - b_3)e_4 + (a_6 + b_2)e_5 \\ &(-a_3 + b_5)e_6 + (-a_4 - b_4). \end{aligned}$$

Further observe that there is no wedge product relation involving only e_1, e_2, e_3 and e_4. Hence we are free to pick the action of Xe_4, as long as we respect the anti-symmetry.

Hence:

$$Xe_4 = -a_3e_1 - b_2e_2 + (-a_5 + b_3)e_3 + c_1e_5 + c_2e_6 + c_3e_7.$$

With $c_1, c_2, c_3 \in \mathbf{R}$. Note further that the action of X on e_5, e_6, e_7 is now fully determined by further wedge product relations, most simply $e_1 \wedge e_4 = e_5$, $e_2 \wedge e_4 = e_6$ and $e_3 \wedge e_4 = e_7$. Hence we are now free to write out the most general form of $X \in g_2$.

$$X = \begin{pmatrix} 0 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & -a_6 \\ a_1 & 0 & -b_1 & -b_2 & -b_3 & -b_4 & -b_5 \\ a_2 & b_1 & 0 & -a_5+b_3 & -a_6+b_2 & a_3-b_5 & a_4+b_4 \\ a_3 & b_2 & a_5-b_3 & 0 & -c_1 & -c_2 & -c_3 \\ a_4 & b_3 & a_6+b_2 & c_1 & 0 & -a_1-c_3 & -a_2+c_2 \\ a_5 & b_4 & -a_3+b_5 & c_2 & a_1+c_3 & 0 & -b_1-c_1 \\ a_6 & b_5 & -a_4-b_4 & c_3 & a_2-c_2 & b_1+c_1 & 0 \end{pmatrix},$$

with:

$$a_1, ..., a_6, b_1, ..., b_5, c_1, c_2, c_3 \in \mathbf{R}.$$

Observe that g_2 therefore has fourteen real dimensions, and hence G_2 is also of real dimension fourteen.

We note a further notation that is sometimes of use:
Write A_{ij} for the element of $gl(7)$ which acts on basis elements as:

$$A_{ij}e_k = \begin{cases} e_i \; for \; e_k = e_j \\ -e_j \; for \; e_k = e_i \\ 0 \; otherwise. \end{cases}$$

We note that these elements span $so(7)$, as they generate all anti-symmetric matrices. Hence it is possible to form a basis for g_2, equivalent to the matrix form above. Note that they are the infinitesimal parts of various rotations in $\mathbf{R}^7$. For example :

$$exp(A_{ij}t)x^k e_k = (e_i \cos t + e_j \sin t)x_i + (e_j \cos t - e_i \sin t)x_j.$$

In [15] it is observed that g_2 is spanned by a series of two-dimensional subspaces, for example:

$$aA_{23} + bA_{45} + cA_{76},$$

where $a + b + c = 0$. We observe that the relevent co-efficients in the matrix form are:

$$a = b_1,$$

$$b = c_1,$$

$$c = -b_1 - c_1,$$

and so the subspace is indeed in g_2. The full set of subspaces is:

$$aA_{23} + bA_{45} + cA_{76},$$
$$aA_{31} + bA_{46} + cA_{57},$$
$$aA_{12} + bA_{47} + cA_{65},$$
$$aA_{51} + bA_{73} + cA_{62},$$
$$aA_{14} + bA_{72} + cA_{36},$$
$$aA_{17} + bA_{24} + cA_{53},$$
$$aA_{61} + bA_{34} + cA_{25},$$

where in each subspace $a+b+c=0$. We have obtained a set of two-dimensional subspaces of g_2, which form the whole of g_2 under the direct sum. This decomposition is useful shorthand for looking at specific g_2 elements, as occurs in [15] where it is used extensively in identifying subgroups of G_2 and the associated sub-algebras of g_2.

Part IV

CR Submanifolds in Almost Hermitian Manifolds

We start the investigation of CR submanifolds with the case of almost Hermitian manifolds as ambient spaces. This is the most general case that we shall consider, as any almost complex manifold may be given a Hermitian metric, and in practice many manifolds under investigation have Hermitian structure with their usual metrics. It is also a common feature of Kaehler and nearly Kaehler manifolds which we will examine in greater detail. Hence results in this chapter will show common theorems between these two types, where as later we shall indicate the differences in theorems.

While a relatively weak condition, the Hermitian property gives some interesting general results - often neglected, or not noted as such when developed for more specialised results. For example in [23] Sekigawa produces results for the specific case of S^6, which are actually of more general application. It is a primary use of reviews of this type to isolate results and indicate their position in the more general theory. Many of the given results may be found in Bejancu [3].

Throughout this chapter we shall generally be considering $\tilde{M}$ as an almost Hermitian manifold, with submanifold M. We denote the complex structure by J, and decompose into tangential and perpendicular components as before (cf eq. ??eqno3), and (3.2)) - for U a tangential vector field to M, ξ perepndicular to M:

$$JU = \phi U + \omega U,$$

$$J\xi = B\xi + C\xi,$$

where $\phi U, B\xi$ are tangential to M, and $\omega U, C\xi$ are perpendicular to M.

In cases where M is a CR submanifold we will use P and Q to represent projection of TM onto D and $D^\perp$ respectively.

6.0.3 Theorem

M is a CR submanifold if and only if

a. $\mathrm{rank}(\phi)$ = constant, and

b. $\omega \cdot \phi = 0$

Proof Firstly assume that M is a CR submanifold. We see that:

$$\phi X = JPX,$$

$$\omega X = JQX, \ \forall X \in \Gamma(TM).$$

Hence $\mathrm{rank}(\phi)$ is a constant, as it is equal to $\mathrm{rank}(JP)$, and by definition of a CR submanifold P is of constant rank. If we combine these two expressions, then we may write:

$$\omega \cdot \phi X = JQ(JPX) = 0,$$

as $JPX \in D$.

Conversely suppose conditions a. and b. are satisfied. We then define D by $D = Image(\phi)$. Observe now that if $X \in \Gamma(D)$, then $X = \phi Y$ for some $Y \in \Gamma(TM)$. Then:

$$JX = J\phi Y = \phi^2 Y + (\omega \cdot \phi)Y = \phi^2 Y \in Image(\phi) = \Gamma(D).$$

Hence D is indeed preserved by J. We now define $D^\perp$ as the bundle perpendicular to D in TM. For an arbitrary $Y \in \Gamma(TM)$ we can write $Y = U + Z, U \in \Gamma(D), Z \in \Gamma(D^\perp)$, and $X \in \Gamma(D^\perp)$, then:

$$\begin{aligned} < JX, Y > &= < JX, U + Z > \\ &= - < X, JU + JZ > \\ &= - < X, JZ > \\ &= - < X, \phi Z > \\ &= 0. \end{aligned}$$

So we see that $JD^\perp$ is indeed perpendicular to M, and so M is truly a CR submanifold as reuired.//

We can prove a related theorem:

6.0.4 Theorem

M is a CR submanifold if and only if
a. rank(B) = constant, and
b. $\phi \cdot B = 0$

Proof In the case that M is a CR submanifold, we do not define B explicitly. We note that Image$(B_p) \subset D_p^\perp$ at each point $p \in M$. To see this consider:

$$< BV, Y >=< JV, Y >= - < V, JY >= 0,$$

for $V \in \Gamma(TM^\perp)$ and $Y \in \Gamma(D)$.On the other hand if we take $U \in \Gamma(D^\perp)$ so that $JU \in \Gamma(TM^\perp)$, then we see that:

$$-U = J^2U = BJU + CJU = BJU.$$

Hence $D^\perp \subset Image(B)$, and so $D^\perp = Image(B)$, and so B is of constant rank. Further we may write:

$$JBV = \phi BV + \omega BV.$$

And we deduce that $\phi \cdot B = 0$, as both sides are perpendicular to TM.

Conversely suppose that conditions a. and b. hold. Then we define $D^\perp$ as the image of B, and D as the distribution perpendicular to $D^\perp$ in TM. It is simple to show that $D^\perp$ maps to $TM^\perp$ under J - take $X \in \Gamma(D^\perp), Y \in \Gamma(TM)$ then, for some $V \in \Gamma(D^\perp)$:

$$< JX, Y >=< JBV, Y >=< \phi \cdot BV, Y >=< 0, Y >= 0.$$

Further we demonstrate that JD is perpendicular to both $TM^\perp$ and $D^\perp$, and hence is preserved - for $X \in \Gamma(D), Y = BV \in \Gamma(D^\perp), Z \in \Gamma(T^\perp)$, then

$$< JX, Y >= - < X, JY >=< X, JBV >= - < X, \phi \cdot BV >= 0,$$

$$< JX, Z >= - < X, JZ >= - < X, BZ >= 0.$$

Hence D, $D^\perp$ are as required, and M is a CR submanifold.//

Note in both of these theorems we needed the almost Hermitian structure to show that certain spaces are perpendicular. Without the Hermitian structue we have no information concerning the way in which J relates to the metric, and hence no manipulations are available between equations involving the metric. We may compare this with the way in which Kaehler and nearly-Kaehler structures give methods for moving the complex structure through covariant differentiation. We note that there is very little work on CR submanifolds (or indeed more generally) in cases of non-Hermitian manifolds.

We note in passing that for a CR submanifold M, ϕ has the properties:

$$\phi^2 = -P,$$

and

$$\phi^3 + \phi = 0,$$

and so by defintion ϕ is an **f-structure** on the bundle TM. Similarly C forms an f-structure on the tangent space perpendicular to TM in $\tilde{M}$. An f-structure may be defined by these two properties, and there is a body of work on such structures, and the reader is referred to Bejancu[5] for how these ideas may be applied to CR submanifolds. One may naively consider them as an analogu of amost complex structures on odd dimensional submanifolds, an idea which may be made more precise. We shall not go into further details of f-structure here.

Recall that we extended the Nijenhuis tensor $N = [J, J]$, to aribtrary linear maps on vector fields such as ϕ by:

$$[\phi, \phi](X, Y) = [\phi X, \phi Y] + \phi^2[X, Y] - \phi[X, \phi Y] - \phi[\phi X, Y],$$

and similarly for ω, B, C.

It is then useful to write the relation:

$$N(U, V) = [\phi, \phi](U, V) - Q[U, V] - \omega([\phi U, V] + [U, \phi V]), \tag{6.1}$$

for $U, V \in \Gamma(D)$.

Proof By direct substitution.//

We consider some examples of integrability conditions for the distributions D and $D^{\perp}$.

6.0.5 Theorem

A CR submanifold M has D integrable if and only if

$$N(U,V)^{T} = [\phi,\phi](U,V),$$

for all $U, V \in \Gamma(D)$.

Proof Take tangential parts of the identity (??eq29)). If D is integrable then the terms $[\phi U, V], [U, \phi V]$ are contained in D, and so ω of these is zero, leaving the indeitiy required. The converse follows similarly. //

Several other integrability conditions may be demonstrated for D, but these are of similar character to those already given. As they will not be used for further proofs we refer the reader to chapter 2 of Bejnacu's work[3].

As contrast we will prove an integrability result for the real distribution $D^{\perp}$.

6.0.6 Theorem

Let M be a CR submanifold of an almost Hermitian manifold, with real distribution $D^{\perp}$. Then $D^{\perp}$ is integrable if and only if the tensor $[\phi,\phi]$ vanishes identically on $D^{\perp}$.

Proof For $X, Y \in \Gamma(D^{\perp})$ then we may write:

$$[\phi,\phi](X,Y) = -P[X,Y],$$

and the result follows. The identity may be demonstrated by substitution

$$\begin{aligned} [\phi,\phi](X,Y) &= [\phi X, \phi X] + \phi^2[X,Y] - \phi[X,\phi Y] - \phi[\phi X, Y] \\ &= \phi^2[X,Y] \\ &= -P[X,Y]. \end{aligned}$$

Recalling that ϕ acts only on vector fields in $\Gamma(D)$. //

We note in passing that the analogous result holds for D in the case of Hermitian manifolds - simply observe that N is identically zero for Hermitian manifolds.

It is also worth noting that when the ambient manifold is given a Kaehler structure both of these theorems may be strengthened considerably - in fact we shall see that in Kaehler manifolds the distribution $D^{\perp}$ is *always* integrable.

6.1 Parallel Distributions and ϕ -connections

We consider now some results proven in general by Bejancu in [4], the main result being that CR submanifolds with ϕ connections are necessarily CR product submanifolds (see 6.1.2 for defintition). Although proved by Bejancu for CR submanifolds in almost Hermitian submanifolds it is used extensively by Chen[10] with respect to Kaehler manifolds, who developed the theorem in the case of ambient manifolds with Kaehler structure independantly. Chen gives an alternative proof (of a weaker theorem) for Kaehler manifolds which will be indicated in the relevant chapter.

A distibution K is **parallel** with respect to some connection ∇ if:

$$\nabla_X Y \in \Gamma(K) \ \ \forall Y \in \Gamma(K),$$

and X an arbitrary vector field on the relevant manifold. We may then show that:

6.1.1 Thoerem

Given a CR submanifold M, in an almost Hermitian manifold $\tilde{M}$ with Levi-Civita connection $\tilde{\nabla}$, the distribution D and $D^{\perp}$ are parallel with respect to ∇ if and only if D and $D^{\perp}$ are both integrable distributions, with leaves totally geodesic in M.

Proof Suppose D and $D^{\perp}$ are paralle with respect to ∇. Then as ∇ is torsion free, for $U, V \in \Gamma(D)$ and $X, Y \in \Gamma(D^{\perp})$ we may write:

$$[U, V] = \nabla_U V - \nabla_V U \in \Gamma(D),$$

$$[X, Y] = \nabla_X Y - \nabla_Y X \in \Gamma(D^{\perp}).$$

And hence by Frobenius theorem D, $D^{\perp}$ are integrable distributions. Now take M^c some leaf of D, and consider the second fundamental form h^c of M^c in M, then:

$$h^c(U, V) = \tilde{\nabla}_U V - \nabla^c_U V,$$

where $U, V \in \Gamma(TM^c)$ and ∇^c is inherited connection on M^c. However observe that the first term is parallel to TM^c as D is a parallel distribution, and h^c is defined as the perpendicular part of $\nabla_U V$ hence is identically zero. Hence M^c is geodesic in M. Similary for any arbitrary leaf of $D^{\perp}$.

Conversely suppose that D, $D^{\perp}$ are integrable distributions, and the leaves are all totally geodesic in M. As the leaves are totally geodesic the second fundamental form of each leaf is zero. Hence for $U, V \in \Gamma(TM^c)$, for some leaf M^c of D, the covariant derivatives $\tilde{\nabla}_U V$ has no part perpendicular to TM^c, by similar arguments as those above. Hence:

$$\tilde{\nabla}_U V \in \Gamma(D),\ \forall U, V \in \Gamma(D),$$

and by identical arguments:

$$\tilde{\nabla}_X Y \in \Gamma(D^\perp),\ \forall X, Y \in \Gamma(D^\perp).$$

It remains to demonstrate that $\tilde{\nabla}_X V \in \Gamma(D)$, and $\tilde{\nabla}_U Y \in \Gamma(D^\perp)$, however:

$$\begin{aligned} 0 &= \tilde{\nabla}_X < Y, V > \\ &= < \tilde{\nabla}_X Y, V > + < Y, \tilde{\nabla}_X V > \\ &= < Y, \tilde{\nabla}_X V > . \end{aligned}$$

Hence $\tilde{\nabla}_X V \in \Gamma(D)$ and similarly we show that $\tilde{\nabla}_U Y \in \Gamma(D^\perp)$. And so D, $D^\perp$ are both parallel with respect to $\tilde{\nabla}$ and the proof is complete.//

We recall that if M has distributions D, $D^\perp$ with integrable leaves, each leaf geodesic in M then M has a product manifold structure, and further M is a CR product, and hence derive the following corollary.

6.1.2 Corollary

Given a CR submanifold M, in the manifold $\tilde{M}$ with Levi-Civita connection $\tilde{\nabla}$, then the distribution D and $D^\perp$ are parallel with respect to ∇ if and only if M is a CR product submanifold.

We will call a connection ∇ on a CR submanifold M a **ϕ-connection** if:

$$\nabla_X \phi = 0 \forall X \in \Gamma(TM),$$

where ϕ is the tangential component of the complex structure J on the CR submanifold M as usual. We prove the following useful theorem:

6.1.3 Theorem

If M is a CR submanifold of an almost Hermitian manifold $\tilde{M}$, and the Levi-Civita connection ∇ on M is a ϕ-connection then M is a CR product submanifold.

Proof For the proof we require the following lemma:

6.1.4 Lemma

All ϕ-connections on M take the form:

$$\nabla_X Y = P\nabla'_X PY + Q\nabla'_X QY + \frac{1}{2}((\nabla'_X \phi)\phi Y + PK(X, PY) - \phi K(X, \phi Y)) + QS(X, QY),$$

for all $X, Y \in \Gamma(TM)$. Where ∇' is a connection with respect to which D, $D^{\perp}$ are parallel, and K, S are arbitrary vector fields of type (1, 2) on M. The proof of the lemma is of reasonable length, and does not provide great insights into the CR structure of the submanifold. The reader is referred to Theorem 3.1 in Chapter 2 of Bejancu[3] for the full proof.

Given this Lemma, we quickly observe that for any ϕ-connection, D and $D^{\perp}$ are parallel with respect to the Levi-Civita connection, and hence by the previous theorem M is a CR product submanifold.//

A more direct proof, of a slightly weaker theorem for the case of Kaehler manifolds will be demonstrated in the chapter on Kaehler manifolds, not purely out of interest but also because in the course of the proof several interesting results are demonstrated. Further although slightly weaker in requirements, we shall see that for Kaehler manifolds we can obtain a converse theorem, lacking here.

Part V

Some Results for CR Submanifolds in Nearly-Kaehler Manifolds

We include some results here, partly because they follow naturally from the previous work on almost Hermitian spaces, and partly because we can make inferences to the Kaehler case, which will be considered in the next chapter.

If $\tilde{M}$ is a nearly-Kaehler manifold, $X, Y \in \Gamma(D)$, we have the following identity:

$$\begin{aligned}[JX, Y] + [X, JY] &= \frac{1}{2}J([J, J](X, Y)) + J[X, Y] + \\ &\quad \nabla_{JX}Y - \nabla_{JY}X + h(JX, Y) - h(X, JY),\end{aligned} \tag{6.2}$$

which we obtain by writing out the expressions in the usual way, observing the anti-symmetry of the Nijenhuis tensor, and using the nearly-Kaehler property that:

$$(\tilde{\nabla}_X J)Y + (\tilde{\nabla}_Y J)X = 0,$$

to eliminate further terms. It is further possible to simplify the two brackets on the left hand side, and combine with the terms in ∇ as ∇ is a torsion free connection. We use this identity to prove the following three results, the first ofwhich is due to Sato [21].

6.1.5 Theorem

The distribution D is integrable if and only if : $h(JU, V) = h(U, JV)$ and $N(U, V) \in D$ for $U, V \in D$.

Proof We refer to Eq (6.2) and deduce that if D is integrable then:

$$h(U, JV) - h(JU, V) = \frac{1}{2}J([J, J](U, V)).$$

We obtain this result by taking only terms perpendicular to D, and applying Frobenius theorem as usual.//

We have already shown that when D is integrable and $\tilde{M}$ is an almost Hermitian manifold (Theorem 6.0.5) we have:

$[J, J] = [\phi, \phi]$ on D, $[J, J]^{\perp} = 0$ on D and $Q[\phi, \phi] = 0$ on D.

Hence : $h(U, JV) - h(JU, V) = 0$, and the two required statements hold.

Conversely, if the conditions hold then from equation (30) we may write:

$$J[U, V] = \nabla_U JV - \nabla_V JU - \frac{1}{2}J[J, J](U, V).$$

We now take some $Z \in \Gamma(D^{\perp})$, (and consequently JZ is perpendicular to M), and so:

$$< [U, V]Z >= - < J[U, V], JZ >= 0,$$

where we have noted that the terms in ∇ are parallel to M and the term in $[J,J]$ are in $\Gamma(D)$ by our assumption. Hence $[U,V] \in \Gamma(D)$, (it is necessarily in $\Gamma(TM)$, by virtue of M being a submanifold, hence its tangent bundle is integrable), and by Frobenius theorem D is an integrable distribution.//

We require a Lemma, which is deduced in Bejancu[3], and is also derived in the course of Sekigawa's work[23].

6.1.6 Lemma

The Nijenhuis tensor on a nearly Kaehler manifold may be written as:

$$[J,J](X,Y) = 4J(\tilde{\nabla}_Y JX).$$

Proof Firstly we note that we are using a torsion free tensor, amd so:

$$[X,Y] = \tilde{\nabla}_X Y = \tilde{\nabla}_Y X.$$

Further we have the nearly Kaehler structure, $(\tilde{\nabla}_X J)X = 0$. If we apply this to vector field $X+Y$, we see that:

$$(\tilde{\nabla}_{X+Y} J)(X+Y) = 0,$$

and hence

$$(\tilde{\nabla}_X J)Y + (\tilde{\nabla}_Y J)X = 0.$$

From this we derive the relation:

$$\begin{aligned}
(\tilde{\nabla}_{JY} J)X &= -(\tilde{\nabla}_X J)(JY) && (6.3)\\
&= \tilde{\nabla}_X Y + J(\tilde{\nabla}_X JY) && (6.4)\\
&= J((\tilde{\nabla}_X J)Y). && (6.5)
\end{aligned}$$

Now we may write out the Nijenhuis tensor in full, and use these relations to simplify:

$$\begin{aligned}
[J,J](X,Y) &= (\tilde{\nabla}_{JX} J)Y - (\tilde{\nabla}_{JY} J)X + J((\tilde{\nabla}_Y J)X && (6.6)\\
&\quad -J((\tilde{\nabla}_X J)Y) && (6.7)\\
&= 2(\tilde{\nabla}_Y X + J(\tilde{\nabla}_Y JX) - \tilde{\nabla}_X Y - J(\tilde{\nabla}_x JY)) && (6.8)\\
&= 2J((\tilde{\nabla}_Y JX - J(\tilde{\nabla}_Y X)) - (\tilde{\nabla}_X JY - J(\tilde{\nabla}_X Y))) && (6.9)\\
&= 2J((\tilde{\nabla}_Y J)X - (\tilde{\nabla}_X J)Y) && (6.10)\\
&= 4J((\tilde{\nabla}_Y J)X). && (6.11)
\end{aligned}$$

This proves the lemma.//

We apply this lemma to obtain the following theorem, originally due to Urbano[25] (the proof follows immediately from the given work)

6.1.7 Theorem

Let M be a CR submanifold of a nearly Kaehler manifold $\tilde{M}$. Then the distribution D is integrable if and only if:

$$(\tilde{\nabla}_U J)Y \in \Gamma(D),$$

and

$$h(U, JV) = h(JU, V).$$

Similar results may be obtained for integrability on $D^{\perp}$, or alternative conditions can be shown for the integrability of D. We do not quote them here however, as (has been remarked) generally nearly-Kaehler manifolds are of restricted interest beyond S^6. The given results are sufficient for a discussion of the consequences for Kaehler manifolds and S^6.

Part VI

CR Submanifolds in Kaehler Manifolds

Chapter 7

General Results for CR submanifolds of Kaehler Manifolds

We now consider the case of CR submanifolds where the ambient manifold $\tilde{M}$ has a Kaehler structure. Recall that this is equivalent to the almost complex structure being parallel :

$$\tilde{\nabla}_X J = 0 \, \forall X \in \Gamma(TM).$$

This definition is of considerable interest due to the fact that many well known spaces are Kaehler with respect to their usual metric - we shall particularly consider complex projective, complex hyperbolic and the flat space $\mathbf{C}^n$. We refer the reader to (for example) Lei, Wolfson[18], chapter 2, where several more complicated Kaehler manifolds are listed - amongst which are some Grassmanian spaces, and quotient spaces, such as $E_7/(E_6 \times T_1)$. We contrast this with the case of nearly Kaehler manifolds, where research is often restricted to S^6.

We shall see that in many cases stronger theorems can be proven for Kaehler manifolds. The key result in this chapter is to relate the curvature of a CR product submanifold to the length of its second fundamental form, and hence place limits on the existence of CR products in some common spaces.

Many of the Kaehler results that we will quote are derived from those in Chen[10]. Where the proof is clear, or fully given in references, we will often omit it, but have given further details at points where non-obvious steps may have been skipped in the original paper, or the proof is of interest for the methods used.

7.1 Some General Results

Here we use standard submanifold theory to derive some relations which hold for CR submanifolds in Kaehler manifolds, and will prove useful in later proofs.

Let $\tilde{M}$ be a Kaehler manifold with complex structure J. We will take M to be a CR submanifold of $\tilde{M}$ with complex distribution D and totally real distribution $D^{\perp}$. We use $\tilde{\nabla}$ as the Levi-Civita connection on $\tilde{M}$, ∇ the inherited connection on M, $\nabla^{\perp}$ the connection on the normal bundle to M, and h the second fundamental form of M.

Recall the Gauss (eq 12) and Weingarten (eq 5) formulae relating these, hence:

$$h(X,Y) = \tilde{\nabla}_X Y - \nabla_X Y, \tag{7.1}$$

$$\tilde{\nabla}_X \xi = -A_\xi X + \nabla^{\perp} \xi, \tag{7.2}$$

where X, Y are in the tangent bundle of M. ξ in the normal bundle to M, and A the shape operator related to h by:

$$< A_\xi X, Y > = < h(X,Y), \xi > . \tag{7.3}$$

As before we shall decompose J into parts tangential and perpendicular to M:

$$\begin{aligned} JX &= \phi X + \omega X, \ X \in \Gamma(TM), \\ J\xi &= B\xi + C\xi, \ \xi \in \Gamma((TM)^{\perp}). \end{aligned}$$

Finally recall that ν indicates the orthogonal subbundle of $JD^{\perp}$ in $T^{\perp}M$, i.e. s.t.

$$T^{\perp}M = JD^{\perp} \oplus \nu, \ JD^{\perp} \perp \nu, \tag{7.4}$$

and so, more fully:

$$T\tilde{M} = D \oplus D^{\perp} \oplus JD^{\perp} \oplus \nu.$$

We now prove some results for general Kaehler manifolds.

First observe that for a Kaehler manifold $\tilde{M}$, M any submanifold we have the result:

$$J\nabla_U Z + Jh(U,Z) = -A_{JZ}U + \nabla^{\perp}_U JZ, \tag{7.5}$$

where Z is in $D^{\perp}$ and U is tangential to M. This follows immediately from equations (2.12) and (6), Gauss and Weingarten, using $\xi = JZ$ in (41) and using torsion free properties of $\tilde{\nabla}$ and the Kaehler property $\tilde{\nabla} J = 0$.

7.1.1 Lemma

Let M be a CR submanifold of Kaehler manifold $\tilde{M}$, then we have the results:

$$< \nabla_U Z, X > = < JA_{JZ}U, X >, \tag{7.6}$$

$$A_{JZ}W = A_{JW}Z, \tag{7.7}$$

$$A_{J\xi}X = -A_{\xi}JX, \tag{7.8}$$

where U is tangential to M, $X \in \Gamma(D)$, $Z, W \in \Gamma(D^{\perp})$, and $\xi \in \Gamma(\nu)$.

The first follows from (7.5) by multiplying by J and then observing that h is normal to M, and for the final term we can write $X = JY$ for some $Y \in \Gamma(D)$:

$$\begin{aligned} < J\nabla_U^{\perp} JZ, X > &= < J\nabla_U^{\perp} JZ, JY > \\ &= < \nabla_U^{\perp} JZ, Y > \\ &= 0. \end{aligned}$$

The second follows in a similar fashion - we may write (7.5) in the two forms:

$$\begin{aligned} J\nabla_W Z + Jh(W, Z) &= -A_{JZ}W + \nabla_W^{\perp} JZ, \\ J\nabla_Z W + Jh(Z, W) &= -A_{JW}Z + \nabla_Z^{\perp} JW, \end{aligned}$$

and then subtract one from the other using the usual equations, and symmetric properties of h to eliminate terms.

The third follows on observing that:

$$\begin{aligned} < A_{\xi}JX, Y > &= < h(JX, Y), \xi > \\ &= < \tilde{\nabla}_Y JX, \xi > \\ &= < J\tilde{\nabla}_Y X, \xi > \\ &= < Jh(X, Y), \xi > \\ &= - < h(X, Y), J\xi > \\ &= < -A_{J\xi}X, Y >, \end{aligned}$$

for arbitray $Y \in \Gamma(TM)$.

7.1.2 Lemma

Let $\tilde{M}$ be a Kaehler manifold, with CR submanifold M. Then for Z, W in $\Gamma(D^{\perp})$

$$\nabla_W^{\perp} JZ - \nabla_Z^{\perp} JW \in JD^{\perp}. \tag{7.9}$$

Proof Take ξ in ν then :

$$\begin{aligned}
< A_{J\xi}Z, W > &= - < \tilde{\nabla}_Z J\xi, W > \\
&= < \nabla^{\perp}_Z \xi, JW > \\
&= - < \xi, \nabla^{\perp}_Z JW > .
\end{aligned}$$

(For the final step we observe that $0 = \nabla^{\perp}_Z < \xi, JW > = < \nabla^{\perp}_Z \xi, JW > + < \xi, \nabla^{\perp}_Z JW >$.)

Hence:

$$< \xi, \nabla^{\perp}_W JZ - \nabla^{\perp}_Z JW > = < A_{J\xi}Z, W > - < A_{J\xi}W, Z > = 0. \tag{7.10}$$

And as this holds for arbitrary $\xi \in \nu$ the result is proved.

We note that as a consequence:

$$\begin{aligned}
J[Z, W] &= J(\nabla_Z W - \nabla_W Z) \\
&= \nabla^{\perp}_Z JW - \nabla^{\perp}_W JZ.
\end{aligned}$$

The second step is obtained by writing down the two identities:

$$J\nabla_W Z + Jh(W, Z) = -A_{JZ}W + \nabla^{\perp}_W JZ,$$

$$J\nabla_Z W + Jh(Z, W) = -A_{JW}Z + \nabla^{\perp}_Z JW,$$

and subtracting one from the other, using the above theormes to remove the unrequired terms. And so by applying this result, and Frobenius theorem to the fact that $D^{\perp}$ is closed under the Lie bracket, we obtain:

7.1.3 Theorem

The totally real distribution $D^{\perp}$ of a CR submanifold M in a Kaehler manifold is integrable.

This result is actually generalizable to **locally conformal Kaehler manifolds**, see[7] for proof and full definition. A locally conformal Kaehler manifold may be defined as a manifold, with an atlas such that the restriction of the the metric to any co-ordinate chart is conformally related to a Kaehler manifold. We contrast this to the nearly-Kaehler, and general almost Hermitian case where such a result is not known. The theorem is **not** extendable to the more general case, as in [7] Blair and Chen give an explicit example of a CR submanifold in a Hermitian manifold with non-integrable $D^{\perp}$.

We also obtain results on the integrability of the complex distribution D.

7.1.4 Theorem

Let M be a CR submanifold of a Kaehler manifold $\tilde{M}$. Then the complex distribution D is integrable if and only if:

$$< h(X, JY), JZ >=< h(JX, Y), JZ >, \tag{7.11}$$

for any X, Y in D, and Z in $D^{\perp}$.

Proof This follows from Theorem 6.1.7 in nearly-Kaehler manifolds. Although the statment is slightly different it is possible to show that the two conditions are equivalent (see further Bejancu[3] Chapter 3).//

7.1.5 Lemma

Let M be a CR submanifold of Kaehler manifold $\tilde{M}$. The leaf $M^{\perp}$ of $D^{\perp}$ is totally geodesic in M if and only if :

$$< h(D, D^{\perp}), JD^{\perp} >= 0. \tag{7.12}$$

Proof This follows from (7.9), see [2]

Recall that the leaves are totally geodesic in a CR product submanifold, hence we have a condition for CR products. We also obtain the following lemma:

7.1.6 Lemma

If M is a CR submanifold of Kaehler manifold $\tilde{M}$, D integrable, and equation (7.12) holds (i.e. the leaves of $D^{\perp}$ are totally geodesic), then for any X in D and ξ in $JD^{\perp}$, we have that:

$$A_{\xi}JX = -JA_{\xi}X. \tag{7.13}$$

Proof This follows very quickly from the above condition that:

$$< h(X, JY), JZ >=< h(JX, Y), JZ > .$$

For $X, Y \in \Gamma(D)$, $Z \in \Gamma(D^{\perp})$.We observe that:

$$\begin{aligned} < h(X, JY), \xi > &= < h(JX, Y), \xi > & (7.14)\\ < A_{\xi}X, JY > &= < A_{\xi}JX, Y > & (7.15)\\ < -A_{\xi}X, Y > &= < A_{\xi}JX, Y >, & (7.16) \end{aligned}$$

and hence the result follows.//

We will explicitly define define a connection ∇ on the forms ϕ, ω, B, C derived from the complex structure J, extending the usual connection ∇ naturally, as follows:

$$(\nabla_U\phi)V = \nabla_U(\phi V) - \phi\nabla_U V, \tag{7.17}$$
$$(\nabla_U\omega)V = \nabla^{\perp}_U(\omega V) - \omega(\nabla_U V), \tag{7.18}$$
$$(\nabla_U B)\xi = \nabla_U(B\xi) - B\nabla^{\perp}_U\xi, \tag{7.19}$$
$$(\nabla_U C)\xi = \nabla^{\perp}(C\xi) - C\nabla^{\perp}_U\xi, \tag{7.20}$$

where U, V are tangential to M, ξ is normal to M.

Recall that we call ϕ parallel if $\nabla\phi = 0$ (and similarly for ω, B, C). We have already shown that if ,for a CR submanifold M, ∇ is ϕ-connection then M is a CR product submanifold (in almost Hermitian ambient manifold $\tilde{M}$). Certainly if ∇ is a ϕ-connection then ϕ is parallel, hence we immediately have the weaker result that if ϕ is parallel (on M) then M is a product submanifold. We shall also be able to prove the converse (which we did *not* have for the almost Hermitian case). We shall therefore find the following Lemma useful:

7.1.7 Lemma

$$(\nabla_U\phi)V = Bh(U, V) + A_{\omega V}U. \tag{7.21}$$

Proof This follows from making the relevant substitutions from the definitions.//

We now prove a key theorem for CR product submanifolds in Kaehler manifolds:

7.1.8 Theorem

Let $\tilde{M}$ be a Kaehler manifold, with a CR submanifold M, then M is a CR product if and only if P is parallel, $\nabla\phi = 0$.

Proof If ϕ is parallel then from Lemma(7.1.7) we have,

$$Bh(U, V) = -A_{\phi V}U, \tag{7.22}$$

for U, V tangential to M. Further if $X \in \Gamma(D)$ then $\phi X = 0$, and so $Bh(U, X) = 0$. Hence by the symmetry of h we have :

$$A_{JZ}X = 0, \tag{7.23}$$

for $X \in \Gamma(D), Z \in \Gamma(D^{\perp})$. Hence by Theorem 7.1.3 and Lemma 7.1.5 , we have the result that $D^{\perp}$ is both integrable and has totally geodesic leaves in M. Finally we consider some leaf M^T of the distribution D, and suppose $X, Y \in \Gamma(TM^T)$, and $Z \in \Gamma(D^{\perp})$, then:

$$
\begin{aligned}
0 &= < A_{JZ}X, Y > && (7.24)\\
&= < JA_{JZ}Y, JX > && (7.25)\\
&= < \nabla_Y Z, JX > && (7.26)\\
&= < Z, \nabla_Y JX >, && (7.27)
\end{aligned}
$$

where we have used equation (45) in the third step, and considering $\nabla_Y < Z, JX >$ in the final step. We deduce that as the tangential connection on M^T (within M) is entirely contained in M^T, then M^T is totaly geodesic in M. Hence combining these results M is a CR product submanifold.

Conversely we suppose that M is a CR product, then for $X \in \Gamma(TM)$ and $U \in \Gamma(D)$, then

$$\nabla_X U \in \Gamma(D).$$

(Using the totally geodesic property of leaves of D). Then we may write:

$$Jh(X, U) = h(X, JU).$$

(Simply observe that

$$h(X, JU) = \tilde{\nabla}_X JU - \nabla_X JU,$$
$$Jh(X, U) = J\tilde{\nabla}_X U - J\nabla_X U,$$

and note that on the right hand side the first times are equal by the Kaehler structure, and the second terms are in $\Gamma(D)$ and hence must be equal as h is perpendicular to M, and Jh is certainly perepndicular to D.)

Hence using the expression for $\nabla\phi$ above we may show that $(\nabla_X\phi)U = 0$:

$$
\begin{aligned}
(\nabla_X\phi)U &= \nabla_X(\phi U) - \phi(\nabla_X U) && (7.28)\\
&= \tilde{\nabla}_X JU - h(X, JU) - J(\tilde{\nabla}_X U - h(X, U)) && (7.29)\\
&= J\tilde{\nabla}_X U - Jh(X, U) - J\tilde{\nabla}_X U + Jh(X, U) && (7.30)\\
&= 0, && (7.31)
\end{aligned}
$$

where we use the fact that $\phi = J$ on D.

Similarly taking $Z \in \Gamma(D^{\perp})$, then $\nabla_X Z \in \Gamma(D^{\perp})$, by leaves of $D^{\perp}$ being totally geodesic, and $(\nabla_X\phi)Z = 0$ (this follows immediatel on observing that $\phi D^{\perp} = 0$). Hence the full result.//

We have in the course of the proof also:

7.1.9 Lemma

A CR submanifold M in a Kaehler manifold $\tilde{M}$ is a CR product if and only if:

$$A_{JD^{\perp}}D = 0.$$

Note We compare this result with the similar result Theorem 7.1.8 for the more general almost hemritian case. We note that the result here is stronger in that we also have the converse result. Although proved by Chen in [10] in the way indicated we note that the proof was previously demonstrated for anti-holomorphic submanifolds (Bejancu, Kon, Yano [2]), and of course the proof in one direction could be inferred from the almost-Hermitian case. It is interesting to note that the converse (that CR product implies that ϕ is parallel) is the more direct part of the proof, but reliant on the Kaehler structure. We also call attention to some notation variation between sources, for example Chen calls ϕ parallel for $\nabla\phi = 0$, but Bejancu[3] calls this a manifold with a ϕ -connection. This is possibly to differentiate from his previous defintion of parallel in the case of the distributions $D, D^{\perp}$, where he only requires that the connection preserves the distribution, rather than being zero.

We now prove a result of great interest, relating the curvature of the ambient manifold to the second fundamental form of a CR product. This immediately has consequences for simple Kaehler manifolds.

7.1.10 Theorem

If M is a CR product submanifold of a Kaehler manifold $\tilde{M}$ then for unit vectors X in D and Z in $D^{\perp}$ then:

$$\tilde{H}_B(X, Z) = 2\|h(X, Z)\|^2, \tag{7.32}$$

where $\tilde{H}_B(X, Z) = \tilde{R}(X, JX; JZ, Z)$ defines the holomorphic bichapteral curvature of X, Z.

Proof Take M to be a CR product, so we are free to use the lemmas above. Recall the Codazzi equation for an embedded submanifold:

$$\tilde{R}(X, JX; Z, JZ) =< \nabla^{\perp}_{X}h(JX, Z) - \nabla^{\perp}_{JX}h(X, Z), JZ > . \tag{7.33}$$

And so using the theorems for Kaehler manifolds we may re-arrange:

$$
\begin{aligned}
\tilde{R}(X, JX; Z, JZ) &= <h(X,Z), \nabla^{\perp}_{JX} JZ> - <h(JX,Z), \nabla^{\perp}_{X} JZ> \\
&= <h(X,Z), J\tilde{\nabla}_{JX} Z> - <h(JX,Z), J\tilde{\nabla}_{X} Z> \\
&= <h(X,Z), J\sigma(JX,Z)> - <\sigma(JX,Z), Jh(X,Z)> \\
&= <h(X,Z), h(X,Z)> + <h(X,Z), h(X,Z)> \\
&= 2\|h(X,Z)\|^2,
\end{aligned}
$$

and hence the result is proved.//

As an immediate consequence we note the following corollary:

7.1.11 Corollary

If $\tilde{M}$ is a Kaehler manifold with positive holomrophic bichapteral curvature, $\tilde{H}_B > 0$, and M a proper CR product. Then M is not anti-holomrophic, and M is not totally geodesic in $\tilde{M}$.

Proof Observe that if M were anti-holomorphic then we would have $\tilde{H}_B = 0$, as in that case $h(X, Z) = 0$ in $\tilde{M}$ as well as M. Further as $h(X, Z) \neq 0$ we deduce that M is not totally geodesic in $\tilde{M}$. (Note that $h(X, Z) = 0$ for the second fundamental for of leaves of $D, D^{\perp}$ in M, and if M were totally geodesic in $\tilde{M}$ then the second fundamental form would be unchanged.)//

We will now consider some specific Kaehler manifolds, and show how the general theory developed can be used.

7.2 Product CR Submanifolds in Complex Hyperbolic Space H^n

For a CR product M in a Kaehler manifold $\tilde{M}$ we have the result that:

$$\tilde{H}_B(X, Z) = 2\|h(X, Z)\|^2,$$

where X is in D and Z in $D^{\perp}$. Hence for any space of negative bichapteral curvature there can be no proper CR products. In particular there are no proper CR products in any complex hyperbolic spaces.

7.3 Product CR Submanifolds in C^m

7.3.1 Theorem

Every CR product M in $\mathbf{C^m}$ is locally the Riemannian product of a complex submanifold, M^T, in a linear complex subspace $\mathbf{C^N}$ and a totally real submanifold, $M^{\perp}$, of a $\mathbf{C^{m-N}}$. i.e.:

$$M = M^T \times M^\perp \subset \mathbf{C^N} \times \mathbf{C^{m-N}} = \mathbf{C^m}.$$

Proof As M is a CR product in $\mathbf{C^m}$ then Theorem 7.1.10 implies that:

$$h(D, D^\perp) = 0. \tag{7.34}$$

Let M^T, $M^\perp$ be integral submanifolds of $D, D^\perp$. We apply a result of Moore[19], so that, given the restrictions on the second fundamental form, we may conclude that $M = M^T \times M^\perp$ is a product of submanifolds in $\mathbf{R^r} \times \mathbf{R^{2m-r}}$, for some r, clearly greater than $dim_R(D)$, and at most $2m - 2dim(D^\perp)$. As M^T is a complex submanifold of $\mathbf{C^m}$ it is possible to choose $\mathbf{R^r}$ to be a complex linear subspace of $\mathbf{C^m}$.

Further if M is antiholomorphic we see that leaves of $D, D^\perp$ must be holomorphic and real hyperplanes in $\mathbf{C}^m$, simply due to dimension restrictions. The anti-holomorphic examples of CR product submanifolds are therefore of extremely simple form.

7.4 Product CR Submanifolds in Complex Projective Space CP^n

In this chapter we shall examine a specific form of proper CR products in complex projective space via a *Segre embedding* which we shall call a *Standard CR Product*. Further we show that these proper CR products have the smallest possible codimension, and that no other CR product has the same codimension.

7.4.1 Definition

Let $\mathbf{CP^m}$ be m-dimensional complex projective space. We recall that $\mathbf{CP^m}$ has constant positive holomorphic chapteral curvature of 4. For real numbers k, p we define a **Segre mapping**:

$$S_{kp} : \mathbf{CP^k} \times \mathbf{CP^p} \to \mathbf{CP^{k+p+kp}},$$

by

$$(z_0, ..., z_k; \eta_0, ..., \eta_p) \to (z_0\eta_0, ..., z_i\eta_j, ..., z_k\eta_p),$$

where $(z_0, .., z_k)$ and $(\eta_0, ..., \eta_p)$ are homogeneous coordinates of $\mathbf{CP^k}$ and $\mathbf{CP^p}$ respectively. Then let $M^\perp$ be a p-dimensional totally real submanifold of $\mathbf{CP^p}$. Then $\mathbf{CP^k} \times M^\perp$ induces a natural CR product in $\mathbf{CP^{k+p+kp}}$ via the mapping S_{kp}.

7.4.2 Definition

Let $M = M^T \times M^\perp$ be a CR product in $\mathbf{CP^m}$. Let $k = dim_{\mathbf{C}}D$, and $p = dim_{\mathbf{R}}D^\perp$. Then M is called a **Standard CR Product** if and only if :
a. $m = k + p + kp$,
b. M^T is a totally geodesic holomorphic submanifold of $\mathbf{CP^m}$.

We see that this definition includes Segre embeddings, although is not exclusively Segre embeddings (we have some freedom in how the real submanifolds $M^\perp$ are embedded in the space). We shall see that the definition is partly redundant in this case as we shall demonstrate that (a.) implies (b.) for CR products.

7.4.3 Lemma

Let M be a CR product in $\mathbf{CP^m}$. Then:

$$\{h(X_i, Z_\alpha)\} i = 1, ..., 2k, \ \alpha = 1, ..., p$$

are orthonormal vectors in ν, where $X_1, ..., X_{2k}$, and $Z_1, ..., Z_p$ are orthonormal bases for D, $D^\perp$ respectively.

Proof Firstly we note that as $\mathbf{C}P^m$ has constant holomorphic bichapteral curvature we can use the theorem:

$$\tilde{H}_B(X, Z) = 2 < h(X, Z), h(X, Z) >,$$

and so

$$|h(X, Z)| = 1.$$

We note that h is defined linearly in $\tilde{\nabla}, \nabla$, and these are in terms linear in their vector field arguments, and so :

$$< h(X_i, Z), h(X_j, X) >= 0, i \neq j.$$

Further we have already shown (Lemma 7.1.9), that for a CR product $A_{JD^\perp} D = 0$.

Hence we see that:

$$< h(X, Z), JY >=< A_{JY} X, Z >= 0,$$

for $Y \in \Gamma(D^\perp)$, and hence $h(X, Z) \in \nu$. If $dim(D^\perp) = 1$ then the proof is complete. If $dim(D^\perp) > 1$ then (again arguing from h being linear in its arguments) we can infer that:

$$< h(X_i, Z_\alpha, h(X_j, Z_\beta) > + < h(X_i, Z_\beta), h(X_j, Z_\alpha) >= 0,$$

for $i \neq j$, $\alpha \neq \beta$.

As M is a CR product we have:

$$< R(X_i, X_j)Z_\alpha, Z_\beta >= 0.$$

We shall quote a result from Blair, Chen[7] (Theorem 6.1), that for a CR product in $\mathbf{C}P^n$ we have:

$$< \tilde{R}(X_i, X_j)Z_\alpha, Z_\beta >= 0.$$

Combining these two curvature results, with the Gauss equation (2.12) we see that:

$$< R(X_i, X_j)Z_\alpha, Z_\beta >=< \tilde{R}(X_I, X_j)Z_\alpha, Z_\beta > + < h(X_i, Z_\beta), h(X_j, Z_\alpha) >$$
$$- < h(X_i, Z_\alpha), h(X_j, Z_\beta >,$$

and so:

$$< h(X_i, Z_\beta), h(X_j, Z_\alpha) >=< h(X_i, Z_\alpha), h(X_j, Z_\beta) > .$$

And hence we see that this expression is actually equal to zero, and so the $h(X_i, Z_\alpha)$ do indeed form an orthonormal basis for the perpendicular space ν.//

As a consequence we immediately see that:

7.4.4 Theorem

Let M be a CR product in $\mathbf{CP^m}$, then

$$m \geq k + p + kp.$$

Indeed as the given Segre mapping gives results where equality holds, we see that this inequality is the best possible.

7.4.5 Theorem

If M is a CR product of $\mathbf{CP^m}$, with $m = k + p + kp$ then M is a standard CR product.

Proof Take M to be a CR product submanifold in $\mathbf{C}P^m$ with $m = k+p+kp$. Take $X, Y, Z \in \Gamma(D), W \in \Gamma(D^\perp)$. Then from the Gauss equation:

$$0 =< \tilde{R}(X, Y)Z, W > + < h(X, W), h(Y, Z) > - < h(X, Z), h(Y, W) >,$$

where we have used the fact that the leaves of $D, D^\perp$ are totally geodesic in M to see that the term in R is zero. Further as $\mathbf{C}P^m$ is a complex space form, we may use the following identity for the curvature (see standard texts for the derivation):

$$\tilde{R}(X,Y)Z = < Y, Z > X - < X, Z > Y + < JY, Z > JX$$
$$- < JX, Z > JY + 2 < X, JY > JZ,$$

and hence $< \tilde{R}(X,Y)Z, W > = 0$.

Hence:

$$< h(X,W), h(Y,Z) > = < h(X,Z), h(Y,W) > .$$

Further if we set $Y = JX$, and using Theorem 7.1.4 to manipulate terms in h, we have:

$$\begin{aligned}
< h(X,Z), h(JX,W) > &= < h(JX,Z), h(X,W) > && (7.35)\\
&= < Jh(X,Z), h(X,W) > && (7.36)\\
&= - < h(X,Z), Jh(X,W) > && (7.37)\\
&= - < h(X,Z), h(JX,W) >, && (7.38)
\end{aligned}$$

and so $< h(X,Z), h(JX,W) > = < h(JX,Z), h(X,W) > = 0$. Further $< h(X,Z), h(X,W) > = 0$.

Hence,

$$\begin{aligned}
0 &= < h(X+Y,Z), h(X+Y,W) > && (7.39)\\
&= < h(X,Z), h(Y,W) > + < h(Y,Z), h(X,W) > . && (7.40)
\end{aligned}$$

Given $< h(X,Z), h(X,W) > = 0$ we see (by dimension counting arguments), that $h(X,Z)$ lies in JD (we have shown that ν is covered completely by $h(D, D^{\perp})$). However recall the earlier condition for CR products (Lemma 7.1.9) that $A_{JD^{\perp}}D = 0$ - we see that this implies $h(X,Z)$ lies in ν.

(To see this we write

$$< h(X,Z), JU > = < A_{JU}X, Z > = 0,$$

for any $U \in \Gamma(D^{\perp})$)

Hence we must have $h(D,D) = 0$. Also as M is a CR product the leaves M^T of D are totally geodesic in M, and as the second fundamental form over D is zero, we see that each M^T is totally geodesic in $\mathbf{C}P^m$. //

Hence we see that for CR submanifolds of minimal codimension the integral submanifolds of the complex distribution D are totally geodesic in $\mathbf{C}P^m$. We may contrast this with the results in $\mathbf{C}^n$, and the non-existence of CR products in hyperbolic space, and S^6 (see later), to see that to be a CR product is a relatively strong condition, and constructing examples of any complexity is not a trivial task.

7.5 Mixed Foliate Submanifolds

We define a CR submanifold of a Kaehler manifold as **mixed foliate** if D is integrable, and $h(D, D^{\perp}) = 0$. The condition on the second fundamental form is sometimes described as defining a **mixed geodesic** CR submanifold. As the definition is given on Kaehler manifolds the distribution $D^{\perp}$ is automatically integrable. It seems sensible that in extending the definition to non-Kaehler ambient spaces, the condition that $D^{\perp}$ is integrable should be added. By analogue with the results obtained for CR products, we obtain the following:

7.5.1 Lemma

Let M be a mixed foliate CR submanifold (in a Kaehler manifold $\tilde{M}$). Then taking $X \in D$ and $Z \in D^{\perp}$, then we have:

$$\tilde{H}_B(X, Z) = -2\|A_{JZ}\|^2.$$

Proof Firstly note that the mixed foliate condition gives us the following:

$$h(D, D^{\perp}) = 0,$$
$$[D, D] \subset D,$$
$$h(X, JY) = h(JX, Y),$$

for arbitrary $X, Y \in \Gamma(D)$. (The last is a consequence of integrability of D that we have obtained for almost-Hermitian manifolds, Theorem (6.1.7)).

We make substitutions into the Codazzi equation:

$$\begin{aligned}
\tilde{H}_B(X, Z) &= \ < \tilde{R}(X, JX)JZ, Z > & (7.41)\\
&= \ - < \tilde{R}(X, JX)Z, JZ > & (7.42)\\
&= \ - < (\nabla_X h)(Z, JX) - (\nabla_{JX} h)(X, Z), JZ > & (7.43)\\
&= \ - < -h(\nabla_X Z, JX) - h(Z, \nabla_X JX) + h(\nabla_{JX} X, Z) & (7.44)\\
&\quad +h(X, \nabla_{JX} Z), JZ > & \\
&= \ < h(\nabla_X Z, JX) - h(X, \nabla_{JX} Z) + h([X, JX], Z), JZ > & (7.45)\\
&= \ < h(\nabla_X Z, JX), JZ > - < h(X, \nabla_{JX} Z), JZ > & (7.46)\\
&= \ < A_{JZ} JX, \nabla_X Z > - < A_{JZ} X, \nabla_{JX} Z > & (7.47)\\
&= \ < A_{JZ} JX, JA_{JZ} X > - < A_{JZ} X, JA_{JZ} JX > & (7.48)\\
&= \ - < JA_{JZ} X, JA_{JZ} X > - < A_{JZ} X, -J^2 A_{JZ} X > & (7.49)\\
&= \ -\|A_{JZ} X\|^2, & (7.50)
\end{aligned}$$

as required, where we have also used the identities in Lemma 7.1.1.//

We immediately have the following theorem:

7.5.2 Theroem

Let $\tilde{M}$ be some Kaehler manifold with strictly positive holomorphic bichapteral curvature, then there are no mixed foliate CR submanifolds of $\tilde{M}$.

In particular we see that $\mathbf{C}P^m$ admits no mixed foliate CR submanifolds, a result reached independantly by Bejancu, Kon and Yano [2], who proved the more specific result that any complex space form of positive bichapteral curvature admits no mixed foliate CR subamnifold.

Chen[10] also call attention to the fact that geodesic spheres of $\mathbf{C}P^m$ are real hypersurfaces, with $h(D, D^\perp) = 0$, where we take $D^\perp$ to be the tangent bundle over the sphere, and D may be chosen arbitrarily in the tangent space perpendicular to the sphere. Hence as any CR submanifold with real part a totally geodesic sphere is mixed foliate - which we have shown cannot hold. Hence there are no CR submanifolds in complex projective space with real part totally geodesic spheres.

We finally prove a theorem for $\mathbf{C}^m$.

7.5.3 Theorem

Let M be a CR submanifold in $\mathbf{C}^m$, then M is mixed foliate if and only if M is a CR product.

Proof Let M be some CR submanifold in $\mathbf{C}^m$. If M is mixed foliate, then :

$$0 = \tilde{H}_B(X, Z) = -2\|A_{JZ}X\|^2.$$

Hence, $A_{JZ}X = 0$, for all $X \in D, Z \in D^\perp$. However we have already shown (Lemma 7.1.9) that this is a necessary and sufficient condition for M to be a CR product.

Conversely if M is a CR product we already have an expression for $\tilde{H}$ for CR products:

$$0 = \tilde{H}_B(X, Z) = 2\|h(X, Z)\|^2,$$

with $X \in \Gamma(D), Z \in \Gamma(D^\perp)$. Hence $h(X, Z) = 0$, and so M is indeed mixed foliate (the integrability of D is part of the definition for CR product).//

Again Bejancu, Kon and Yano[2] developed the above theorem independantly for antiholomorphic CR submanifolds.

It is interesting to note that the existenc of CR products and CR mixed foliate submanifolds mirror each other, for spaces of positive bichapteral curvature (eg projective spaces) CR products are admitted but not mixed foliates, in manifolds of negative bichapteral curvature (eg hyperbolic spaces) CR products are not admitted but mixed foliates are. Finally in spaces of zero holomorphic bichapteral curvature (eg $\mathbf{C}^m$) the two definitions coincide.

7.6 Some Comments on Further Work on Kaehler Manifolds

Also of interest are several results derived by Chen placing bounds on the length of the second fundamental form. We shall only give the simplest result for $\mathbf{C}P^m$, but refer the reader to the referenced papers for further work on the same subject. We pick out a result concerning the length of the second fundamental form, which follows quickly from the work already given:

7.6.1 Theorem

Let M be a CR submanifold of $\mathbf{C}P^m$. Then the length of the second fundamental form is bounded:

$$\|h\|^2 \leq 4kp,$$

where $k = dim_C D$ and $p = dim_R D^\perp$. If equality holds then leaves of $D, D^\perp$ are totally geodesic in $\mathbf{C}P^m$.

Proof Firstly observe that $\mathbf{C}P^m$ is of constant holomorphic bichapteral curvature 4, and as a consequence:

$$\|h(X, Z)\| = 1,$$

for $X \in \Gamma(D)$ and $Z \in \Gamma(D^\perp)$. Further we have shown that we may write orthonormal bases for $D, D^\perp$ of $\{X_1, ..., X_{2k}\}, \{Z_1, ..., Z_p\}$, with $h(X_A, Z_\alpha) = 0$. Hence, using the linearity of the second fundamental form, we may write:

$$\|h\|^2 = 4kp + \sum_{A,B} \|h(X_A, X_B)\|^2 + \sum_{\alpha,\beta} \|h(Z_\alpha, Z_\beta\|^2,$$

where we have been able to evaluate the cross terms $h(X, Z)$ as 1. Hence we have the required inequality, and immediately observe that equality holds only if $h(D, D) = 0$ and $h(D^\perp, D^\perp) = 0$, which may be shown as equivalent to the integral submanifolds of the distributions being totally geodesic in the ambient manifold.//

We end this chapter by noting that Chen[10] pursues similar lines of argument to deduce further relationships between the second fundamental form, the curvatures and the CR structure of submanifolds.

Part VII

Some Morse Theory Results on CR Submanifolds

The results up to this point have been concerned with the traditional submanifold properties of CR submanifolds, such as the curvature properties, Riemannian product structure, and properties of the second fundamental form. As a contrast we will consider some results obtained by Lei Ni and Jon Wolfson in [18], relating to the application of some Morse theory results and consequences for the topology of CR submanifolds. The work is an extension of techniques applied by Schoen and Wolfson in [22] to complex submanifolds of a Kaehler manifold.

We shall quote several required results, which although far from obvious and of interest in themselves, lie outside the context of the current work.

In [18] work is exclusively carried out for ambient manifolds having Kaehler structure (and some further restrictions), and for the real dimension of the real distribution $D^{\perp}$ to be equal to 1. The restrictions permit the use of some quoted thereoms, and manipulations. Some consideration of the problems encountered for extending the ideas to more general cases are given.

7.7 Some Definitions

Throughout this part we shall consider the ambient space $\tilde{M}$ to be an irreducbile compact Kaehler manifold, complex dimension v of non-negative holomorphic bichapteral curvature.

We define a symmetric bilinear form:

$$H_Y(W, Z) =< \tilde{R}(Y, JY)W, JZ >,$$

for vector fields W, Y, Z over $\tilde{M}$. By the general symmetries of the curvature tensor, for fixed $Y \neq 0$, observe that H_Y is positive semi-definite. The null space of H_Y is then denoted N_Y, and we write $l(Y)$ for the complementary dimension of N_Y. We define the **complex positivity** of the manifold $\tilde{M}$ as :

$$l = \frac{inf}{Y \neq 0} l(Y).$$

We will simply quote the result that l is a well defined on the manifold, and is independent of the evaluation point chosen. Further we note that values of l have been computed for various spaces, and of particular interest to us is the fact that the complex projective space $\mathbf{C}P^v$ has complex positivity v.

We refer the reader to Part I for definitions of the Levi form, and the nullity as defined for a CR submanifold M. We shall assume the the real distribution $D^{\perp}$ over M has dimension 1, and the distribution D complex dimension p.

7.8 Critical Paths Between a Complex Submanifold and a CR Submanifold

Let the ambient space $\tilde{M}$ be an irreducible compact Kaehler manifold with complex positivity l and complex dimension v. Let M be a smooth compact CR submanifold of real dimension $(2p+1)$, and N a compact complex submanifold of complex dimension n.

We consider the space $\Omega(\tilde{M}; M, N)$ consisting of piecewise smooth paths $\gamma : [0,1] \rightarrow \tilde{M}$ s.t. $\gamma(0)$ is on M, and $\gamma(1)$ on N. We define an energy function on these paths by:

$$E(\gamma) = \int_0^1 \|\dot{\gamma}\|^2. \tag{7.51}$$

It may be shown that γ is a critical point of E if:

a. γ is a smooth geodesic in $\tilde{M}$,

b. γ is normal to M at $\gamma(0)$ and to N at $\gamma(1)$.

This result may be obtained by considering some first variation of the energy function:

$$\begin{aligned}
\frac{d}{ds}E(\gamma) &= \int_0^1 \frac{d}{ds} < \dot{\gamma}, \dot{\gamma} > dt & (7.52)\\
&= \int_0^1 \frac{d}{dt} < \frac{d}{ds}\gamma, \dot{\gamma} > - < \frac{d}{ds}\gamma, \frac{d^2}{dt^2}\gamma > dt, & (7.53)
\end{aligned}$$

and observing that the second term is zero if condition (a) holds, and the first part is zero if condition (b) holds.

We are interested in the index of such critical paths in order to apply Morse theory ideas, and hence need to consider the second variation of the energy function. We observe that the second variation of E may be put into the form:

$$\begin{aligned}
\frac{1}{2}E_{**}(W_1, W_2) &= < \nabla_{W_1} W_2, \dot{\gamma} > \Big|_0^1 + \int_0^1 < \nabla_{\dot{\gamma}} W_1, \nabla_{\dot{\gamma}} W_3 > dt \\
&\quad - \int_0^1 < \tilde{R}(\dot{\gamma}, W_1)\dot{\gamma}, W_2 > dt.
\end{aligned}$$

We now need to determine the subspaces for which E_{**} is negative. We will prove the following:

7.8.1 Theorem

Let M be a CR submanifold in $\tilde{M}$, let N be a complex submanifold of $\tilde{M}$, subject to the restrictions above. Let M be of real dimension $2p+1$ and $\tilde{M}$ have complex positivity l. If the nullity of M is everywhere greater than or equal to r, (where necessarily $0 < r \leq p$), then the index of a critical path γ of the energy function E is *at least* $l + 1 - (v - r) - (v - n)$.

The proof of this result is surprisingly straightforward when the necessary manipulations are shown, being in essence a dimension counting problem. The chief interest is how this result may be combined with more general results to generate topological theorems.

Proof Given the situation above let γ be a critical path of $E : \Omega \to \mathbf{R}$. Let M have nullity everwhere greater than or equal to r. Then for any point on M we may pick local vectors $X_1, ..., X_r$ lying in the null space of the Levi form. Further it is possible to select X_i such that $X_i = W_i - iJW_i$, where $W_1, ...W_r, JW_1, JW_r$ are orthonormal vectors. We construct such a basis at $\gamma(0)$.

We extend the vectors W_i at $\gamma(0)$ to vector fields $W_i(t)$, along the whole of γ, by parallel transport. Observe that the vector fields W_i so defined necessarily define varitions of the the path γ. We have chosen W_i to lie in $T_{\gamma(0)}M$, so variations keep $\gamma(0)$ lying in M. However the construction does not necessarily mean that $W_i(1)$ lies in $T_{\gamma(1)}N$, and so $\gamma(1)$ may under the W_i variation be moved off of N, leading to a path not in Ω. Hence W_i does not necessarily belong to $T_\gamma\Omega$. Because we are only attempting to set a bound on the index it is sufficient to consider the W_i which do belong to $T_\gamma\Omega$.

We consider the second variation of the energy for some such varition W_i:

$$\begin{aligned} \frac{1}{2}E_{**}(W_i, W_i) &= < \nabla_{W_i} W_i, \dot{\gamma} > \Big|_0^1 + \int_0^1 < \nabla_{\dot{\gamma}} W_i, \nabla_{\dot{\gamma}} W_i > dt & (7.54) \\ &\quad - \int_0^1 < \tilde{R}(\dot{\gamma}, W_i), W_i > dt & (7.55) \\ &= < \nabla_{W_i} W_i, \dot{\gamma} > \Big|_0^1 - \int_0^1 < \tilde{R}(\dot{\gamma}, W_i), W_i > dt. & (7.56) \end{aligned}$$

(NB We here use ∇ to indicate the metric connection on $\tilde{M}$, we shall not consider connections on the subamnifolds in this chapter). The second term in the first line is observed to be zero as W_i has been constructed by parallel transportation. Similarly we construct fields JW_i along γ by parallel transportation, and obtain the identity:

$$\frac{1}{2}E_{**}(JW_i, JW_i) =< \nabla_{JW_i} JW_i, \dot{\gamma} > \Big|_0^1 - \int_0^1 < \tilde{R}(\dot{\gamma}, JW_i)\dot{\gamma}, JW_i > dt.$$

However we now use the Kaehler structure to move J through covariant differentiation:

$$\begin{aligned} < \nabla_{JW_i} JW_i, \dot{\gamma} > &= < J(\nabla_{JW_i}), \dot{\gamma} > & (7.57) \\ &= - < \nabla_{JW_i} W_i, J\dot{\gamma} > & (7.58) \\ &= - < \nabla_{W_i} JW_i, J\dot{\gamma} > + < [JW_i, W_i], J\dot{\gamma} > & (7.59) \\ &= - < \nabla_{W_i} W_i, \dot{\gamma} > + < [JW_i, W_i], J\dot{\gamma} > . & (7.60) \end{aligned}$$

Note *This is the only element of the proof of the index of paths that relies on the Kaehler structure of the ambient manifold - for almost Kaehler manifolds a further terms will be introduced in this expression.*

We now show that the second term is zero when evaluated at $\gamma(0)$ and $\gamma(1)$. As γ is a critical path, $\dot{\gamma}$ is perpendicular to M, N at $\gamma(0), \gamma(1)$ respectively. Further since (by construction) $X_i = W_i + JW_i$ lies in the null space of the Levi form then $[JW_i, W_i]$ is perpendicular to $JT_{\gamma(0)}$. Further as γ is a critical path, $\dot{\gamma}$ is perpendicular to M, and hence to $[JW_i, W_i]$. (We have used Frobenius theorem to infer that $[JW_i, W_i]$ lies in $T_{\gamma(0)}M$).

In order to show that $< [JW_i, W_i], J\dot{\gamma} >= 0$ when evaluated at $\gamma(1)$, we use an argument due to Frankel(19). Let V be some analytic curve passing through $\gamma(1)$ with tangent vectors W_i, JW_i. Extend W_i, JW_i to vector fields in some region U of $\gamma(1)$. On this region U, $[JW_i, W_I], J[JW_i, W_i]$ are tangential to V, and so at $\gamma(1)$ are perpendicular to both $\dot{\gamma}$ and $J\dot{\gamma}$.

Hence we may drop the associated terms for the second variation:

$$\frac{1}{2}E_{**}(JW_i, JW_i) = - < \nabla_{W_i} W_i, \dot{\gamma} > \|_0^1 - \int_0^1 < \tilde{R}(\dot{\gamma}, JW_i)\dot{\gamma}, JW_i > dt.$$

Thus we have:

$$\frac{1}{2}E_{**}(W_i, W_i) + \frac{1}{2}E_{**}(JW_i, JW_i) = - \int_0^1 (< \tilde{R}(\dot{\gamma}, W_i)\dot{\gamma}, JW_i >$$

$$+ < \tilde{R}(\dot{\gamma}, JW_i)\dot{\gamma}, JW_i > dt.$$

Rewriting using the usual symmetries of $\tilde{R}$:

$$(E_{**}(W_i, W_i) + E_{**}(JW_i, JW_i)) = - \int_0^1 (< \tilde{R}(\dot{\gamma}, J\dot{\gamma})W_i, JW_i >,$$

and the term on the right handside is (minus) the holomorphic bichapteral curvature of the complex lines $\dot{\gamma} \wedge J\dot{\gamma}$ and $W_i \wedge JW_i$. We see therefore that the second variation is necessarily non-positive (as the holomorphic bichapteral curvature is non-negative) for all W_i, JW_i which define a variation in $T_\gamma\Omega$ - i.e. those where $W_i(1), JW_i(1) \in T_{\gamma(1)}N$. Hence the index of the path γ is at least equal to the number of such W_i which are variations in $T_\gamma\Omega$ for which the holomorphic bichapteral curvature given is psoitive.

We consider the full set of $W_1, ..., W_r, JW_1, ..., JW_r$. Firstly observe that W_i, JW_i are perpendicular to both of $\dot{\gamma}$ and $J\dot{\gamma}$ (from the arguments above). Hence the span of such vectors, say:

$$S = span\{W_1, ..., W_r, JW_1, ..., JW_r\},$$

is an r-dimensional complex vector space, within a $(v-1)$ dimensional subspace of $T_{\gamma(1)}\tilde{M}$. Hence the space $S \cap T_{\gamma(1)}N$ has complex dimension of at least $r + n - (v-1)$. If it is given that $\tilde{M}$ has complex positivity l, then we see that the subspace of $S \cap T_{\gamma(1)}N$ for which the holomorphic bichapteral curvature is strictly positive is $(r + n - (v-1) - (v-l)$. Hence:

$$index(\gamma) \geq r + n - (v-1) - (v-l) = 1 + l - (v-r) - (v-n)),$$

as required.//

Given this result for the index we may derive theorems concerning the topology of the manifolds, for example:

7.8.2 Theorem

Let $\tilde{M}$ be a compact Kaehler manifold of non-negative holomorphic bichapteral curvature, of complex dimension v, and complex positivity l. Let M be a compact CR submanifold on $\tilde{M}$ of real dimension $2k+1$, where dim $D^{\perp} = 1$. Suppose the nullity of the Levi form of M is everywhere greater than or equal to r. Let N be a compact complex submanifold of complex dimension n. Then the homomorphisms induced by inclusion:

$$i_* : \pi_j(M, M \cap N) \to \pi_j(\tilde{M}, N),$$

are isomorphisms for $j \leq n + r - v - (v - l)$ and is a surjection for $j = n + r - v - (v - l) + 1$.

The π_j referred to are relative homotopy groups for the relevant spaces. The proof follows from the results on the index of critical paths combined with some more general results from Morse theory connecting index values with such homotopy groups. The reader is referred to the references (e.g. Schoen-Wolfson[22]) for details, which are not immediately relevant.

7.9 Critical Paths Between Two CR Submanifolds

Lei and Wolfson continue by extending the results in the previous chapter to the case where both M and N are CR submanifolds.

7.9.1 Theorem

Let $\tilde{M}$ be an irreducible compact Kaehler manifold of complex dimension v, nonegative holomorphicbichapteral curvature, and complex positivity l. Let M, N be CR submanifolds of $\tilde{M}$ with real distributions of dimension 1, complex distributions of (real) dimensions $2p, 2q$ respectively. Let M have nullity everywhere greater than or equal to r. Let N have nullity everywhere greater than or equal to s. Then the index of a critical path γ (in the space of paths from M to N) is at least:

$$l + 1 - (v - r) - (v - s).$$

Proof The proof is almost identical to that in the case where N is complex, with some minor variation caused by replacing the complex manifold with a CR submanifold. The idea involved are identical however. //

Results concerning the relevant topologies of the two submanifolds is again possible. In addition we may consider the case where M and N are identified as the same submanifold, and so obtain results concerning a single CR submanifold. For example Lei and Wolfson[18] obtain the result

7.9.2 Theorem

Let $\tilde{M}$ be as above. If $m \geq 2(v+1) - l$ then there are no smooth Levi flat subamnifolds of dimension m in $\tilde{M}$ with compact leaves of D.

Proof The proof is via comparing topological consequences of the index results given here, with further general topological results about submanifolds. The details are not of direct relevance to this work.//

7.10 Conclusions

We have shown how very specific CR structure has been used by Lei and Wolfson to place restrictions on the index of paths in the ambient space. It is a direct extension of work for complex submanifolds, and so it is of interest to see how CR structures contrast with the complex case. Here we observe how the estimates on the index may take into account the Levi form, which is not available in the purely complex case, modifying the estimate considerably.

Observe that the work is restricted to CR submanifolds with real distributions of dimension 1 - however it would be simple to extend to a more general case if the nullity were approriately redefined. The restriction to Kaehler manifolds is more complicated - we observed that relaxing the Kaehler condition would generate extra terms in the expression for the second variation. Naturally restrictions could be placed on these extra terms, but it is unlikely that these would be sensible restrictions to place on a manifold. It is more practical to suppose that the relevant steps could be tested for specific non-Kaehler ambient manifolds, for example S^6.

The work in its entirety is highly abstract, it may be that consideration of specific submanifolds, e.g projective spaces, S^6, in which the form of the curvature and connection are well known, might permit more direct proofs, or more restricitve estimates. It is unclear whether the given estimates are improvable, and it would also be of use to produce examples - the most readily available are those due to the Segre mappings in complex projective spaces.

Part VIII

CR Submanifolds in S^6

Chapter 8

Totally Geodesic CR Submanifolds of S^6

It is a known result that the only totally geodesic submanifolds of the sphere are great spheres, most easily described as the interchapter of the sphere with linear subspaces through the origin. In order to simplify analysis of submanifolds we shall use the fact that we only consider CR submanifolds up to some G_2 variation. Hence we can freely choose the direction of e_1 and e_2, and e_4 as a vector perpendicular to e_1, e_2 and $e_1 \wedge e_2$.

For completeness we also consider totally real and totally complex examples as degenerate examples of CR submanifolds. We will generally consider dimensions to be in terms of real dimensions unles otherwise specified. We shall call the constructed submanifold M as usual. Submanifold which are neither complex, real or CR will be described as unclassified.

8.1 1-Dimensional Examples

Necessarily any one dimensional submanifold is totally real. Just observe that for any vector p:

$$\begin{aligned} < p, Jp > &= < Jp, J^2 p > \\ &= - < Jp, -p > \\ &= - < p, Jp > . \end{aligned}$$

This implies that $< p, Jp >= 0$, and hence specifically any totally geodesic example (i.e. some S^1 about the origin) is totally real.

This follows immediately from the almost-Hermitian structure, and so similarly holds in any Hermitian manifold.

8.2 2-Dimensional Examples

These consist of the interchapters of S^6 with some 3-plane. We can use G_2 action to force this 3-plane to include e_1 and e_2.

If e_3 completes the 3-plane then the submanifold is totally complex.

Suppose instead that the 3-plane is completed by $\mu e_3 + \nu e_4$.

Certainly if $\mu = 0$, $\nu = 1$ then the submanifold is totally real. Again this is essentially unique up to G_2 actions. However if μ is not zero then the submanifold is unclassified, as (for example) at e_1, $Je_2 = e_1 \wedge e_2 = e_3$ is neither contained in, nor perpendicular to $T_{e_1}M = span < e_2, \mu e_3 + \nu e_4 >$.

8.3 3-Dimensional Examples

The most interesting possibility is one with a two (real) dimensional complex distribution D, and a one dimensional real distribution $D^\perp$. Let us take the point e_1 to be on M, select e_2 as being in $T_{e_1}M$ and $D(e_1)$ and so e_3 is also. We are free to select a further perpendicular direction e_4 which is in $T_{e_1}M$ and $D^\perp(e_1)$. However observe that at the point e_4 we have tangent space $< e_1, e_2, e_3 >$, but $Je_1 = e_5$, $Je_2 = e_6$ and $Je_3 = e_7$, and so the manifold has totally real characterisitics at this point. hence the submanifold is of undefined type.

The only further option is a totally real example. Again we look at the point e_1, and can assume e_2, e_4 are in the tangent space at this point. Observe that at e_1, $Je_2 = e_3$, $Je_4 = e_5$ and at the point e_2, $Je_4 = e_6$. Hence we are forced to complete the 4-plane with e_7. Observe that this indeed forms a totally real submanifold (simply notice that e_1, e_2, e_4, e_7 wedge products all have results in e_3, e_5, e_6).

8.4 4-Dimensional Examples

We firstly note that Grey[13] has previosly demonstrated that there are no four (real) dimensional complex submanifolds of S^6. We refer the reader to this reference for the general proof, but give a proof for totally geodesic submanifolds here. Take e_1 to belong the submanifold, and as before we can pick e_2, e_3, e_4 to lie in the tangent space. Further to make the tangent space complex we require $e_5 \in T_{e_1}M$. However if we now look at the point e_4, observe that $Je_2 = -e_6 \notin T_{e_4}M$, and so the submanifold constructed is of undefined type.

It is not possible to construct a totally real example due to the restriction to 6 dimensions - a totally real manifold of dimension 4 would require a minimum of 8 dimensions.

Therefore consider a non-trivial CR example, with 2 (complex) dimensional complex distribution, and 2 (real) dimensional real distribution. We select e_1 as a point on the submanifold, and take $e_2, e_3, e_4 \in T_{e_1}M$, defining a complex distribution $D = span < e_2, e_3 >$, and partially defining the real distribution by $e_4 \in D^\perp$ at this point. Observe that $Je_4 = e_5$, hence $e_5 \perp T_{e_1}M$ and so is perpendicular to the intersecting 5-plane. Further as e_1, e_2, e_3 form an associative plane, at points e_2 and e_3 note that $e_4 \in D' perp$. However at e_2, $Je_4 = e_6$, and at e_3, $Je_4 = e_7$, and so we similarly deduce that e_6, e_7 are both perpendicular to the intersecting 5-plane as well. This means that it is impossible to construct an intersecting 5-plane with the required properties.

Hence all 4-dimensional totally geodesic submanifolds of S^6 are of unclassified type.

8.5 5-Dimensional Examples

We observe that any 5 dimensional manifold M of S^6 is CR, with one dimensional real distribution D, with $D(p) = p \wedge \xi$, for ξ the orthogonal complement of T_pM in T_pS^6.

8.6 Summary of Totally Geodesic CR Submanifolds in S^6

In 1 dimension all submanifolds are totally real.
In 2 dimensions there are unique real and complex submanifolds, plus undefined submanifolds.
In 3 dimensions there is a unique real subamnifold, plus undefined submanifolds.
In 4 dimensions all submanifolds are of undefined type.
In 5 dimensions all submanifolds are (trivially) CR.
(Uniqueness up to G_2 actions.)

Chapter 9

Small Sphere CR subamnifolds in S^6

We continue to the next simplest class of submanifolds in S^6, those of small spheres, formed by the interchapter of S^6 with planes not passing through the origin of $\mathbf{R^7}$.

9.1 1-Dimensional Examples

We have already observe that all 1-dimensional examples are totally real.

9.2 2-Dimensional Examples

Without loss of generality we will take the intersecting plane to be offset perpendicularly by the vector λe_1 with $\lambda \in (0,1)$. (actually $\lambda = 0$ would correspond to the totally geodesic great spheres). We may also choose e_2 to lie in the intersecting plane, and also $\alpha e_3 + \beta e_4$, for some α, β yet to be determined, however not both zero.

Let us first attempt to construct a complex example:

Look at a point $p = \lambda e_1 + \mu e_2$, where μ has been chosen s.t. p indeed lies on S^6. Consider then the result of the tangent space under the complex structure:

$$\begin{aligned} J\alpha e_3 + \beta e_4 &= (\lambda e_1 + \mu e_2) \wedge (\alpha e_3 + \beta e_4) && (9.1) \\ &= \lambda(-\alpha e_2 + \beta e_5) + \mu(-\alpha e_1 + \beta e_6). && (9.2) \end{aligned}$$

Now observe that we require the e_1 component of this to be indentically zero (the tangent space is by construction perpendicular to e_1), hence $\alpha = 0$, and we may take $\beta = 1$. We may therefore rewrite:

$$Je_4 = (\lambda e_1 + \mu e_2) \wedge e_4 \tag{9.3}$$
$$= \lambda e_5 + \mu e_6. \tag{9.4}$$

We deduce that the intersecting plane is spanned by vectore $< e_2, e_4, \lambda e_5 + \mu e_6 >$. However observe then that at other points on the submanifold we will have components in e_1 e.g. generated by $e_4 \wedge e_5$, and hence $\lambda = 0$ contradicting the original assumption. Observe further that this difficulty will arise with constructing any (specifically 4 dimensional) holomorphic submanifold from a small sphere, hence we deduce that there are actually no holomorphic small spheres in S^6.

We now attempty to construct some totally real example: again we look at the point $p = \lambda e_1 + \mu e_2$ and the image of the tangent space under J.

$$J\alpha e_3 + \beta e_4 = (\lambda e_1 + \mu e_2) \wedge (\alpha e_3 + \beta e_4) \tag{9.5}$$
$$= \lambda(-\alpha e_2 + \beta e_5) + \mu(-\alpha e_1 + \beta e_6). \tag{9.6}$$

We see that this is perpendicular to e_2 if and only if $\alpha = 1$, and so again:

$$Je_4 = (\lambda e_1 + \mu e_2) \wedge e_4 \tag{9.7}$$
$$= \lambda e_5 + \mu e_6. \tag{9.8}$$

We deduce that the intersecting plane is spanned by $< e_2, e_4, \mu e_5 - \lambda e_6 >$. However moving to the point $\tilde{p} = \lambda e_1 + \mu e_4$, then :

$$Je_2 = (\lambda e_1 + \mu e_4) \wedge e_2 \tag{9.9}$$
$$= \lambda e_3 - \mu e_6 \tag{9.10}$$

But this is not perpendicular to the vector $\mu e_5 - \lambda e_6$, as by construction λ and μ are both non-zero. Hence we cannot construct a totally real small sphere in 2 dimensions. Observe further that the argument extends to trying to construct totally real small spheres of any dimension.

9.3 3-Dimensional Examples

We have observed that we cannot construct totally real examples of any dimension, and so the only interesting type is CR with D of dimension 2, and $D^{\perp}$ of dimension 1.

As before we will take the intersecting plane to be offset perpendicularly by the vector λe_1 with $\lambda \in (0, 1)$. (actually $\lambda = 0$ would correspond to the totally geodesic great spheres). We may also choose e_2 to lie in the intersecting plane, and also $\alpha e_3 + \beta e_4$, for some α, β yet to be determined, however not both zero. Further we assume $\alpha e_3 + \beta e_4$ to lie in $D(p)$ where $p = \lambda e_1 + \mu e_2$. But by previous arguments we deuce that $\alpha = 0$ and that $D(p)$ is spanned by $span < e_4, \lambda e_5 + \mu e_6 >$.

Now consider a further point $\tilde{p} = \lambda e_1 + \mu e_4$. Observe that D has non-zero interchapter with $span < e_2, \lambda e_5 + \mu e_6 >$. However e_5 will generate elements in e_1, hence we observe that $e_2 \in D(\tilde{p})$. Hence we deduce that the complex part of the tangent space is completed by:

$$
\begin{aligned}
Je_2 &= (\lambda e_1 + \mu e_4) \wedge (e_2) && (9.11)\\
&= (\lambda e_3 - \mu e_6). && (9.12)
\end{aligned}
$$

Hence the intersecting plane must be spanned by $span < e_4, \lambda e_5 + \mu e_6, \lambda e_3 - \mu e_6 >$. Whilst bearing in mind that we must orthonormalize to :

$$span < e_4, \lambda e_5 + \mu e_6, \lambda e_3 - \mu e_6 + \mu^2(\lambda e_5 + \mu e_6 >,$$

(recall that λ, μ are chosen such that $\lambda^2 + \mu^2 = 1$).
Moving back to the point $p = \lambda e_1 + \mu e_2$ observe that :

$$
\begin{aligned}
J\lambda e_3 - \mu e_6 + \mu^2(\lambda e_5 + \mu e_6 &= \lambda(-\lambda e_2 + \mu e_7 + \mu^2(-\lambda e_4 - \mu e_7) + \mu(\lambda e_1 && (9.13)\\
&\quad +\mu e_4 + \mu^2(\lambda e_7 - \mu e_4).
\end{aligned}
$$

This has component in e_4 of:

$$\mu^2(1 - \lambda - \mu).$$

Which is zero if and only if $\lambda + \mu = 1$, but this is incompatible with the construction that $\lambda^2 + \mu^2 = 1$, and both are non-zero. Hence we cannot construct a 3 dimensional CR submanifold which is a small sphere. Observe that the same argument will follow if we attempt to build a higher dimensional CR submanifold with $dim(D) = 2$, $dim(D^{\perp}) > 1$, in each case we are restricted to the choice of D, and this is then incompatible with constructing a totally real $D^{\perp}$.

9.4 4-Dimensional Examples

We have observed that construction of holomorphic examples, and CR examples with $dim(D) = 2$ are not possible. These are the only cases to be considered.

9.5 5-Dimensional Examples

As previously observed all 5-dimensional submanifolds of S^6 are trivially of CR type.

9.6 Summary of Small Sphere CR Submanifolds in S^6

In 1 dimension all submanifolds are totally real.
In 2 dimensions all small spheres are of unclassified type.
In 3 dimensions all small spheres are of unclassified type.
In 4 dimensions all small spheres are of unclassified type.
In 5 dimensions all submanifolds are (trivially) CR.

Hence small spheres are of little interest regarding their CR character.

Chapter 10

CR Product Submanifolds in S^6

We now prove that there are no CR product submanifolds in S^6. This is a result initially due to Sekigawa in [23]. A slightly adapted version occurs in Bejancu's text [3], although the technique is essentially the same. The proof is by contradiction on the relationship of the various associated distributions $D, D^{\perp}, JD^{\perp}$, and ν.

We first of all derive some results:

10.0.1 Lemma

If M is a CR product then D is closed under covariant differentiation in M, i.e. $\nabla_X Y \in \Gamma(D)$ for all $X, Y \in \Gamma(D)$. Similarly $D^{\perp}$ is closed under covariant differentiation in M.

Proof This can be shown by recalling that the integral manifolds of D and $D^{\perp}$ are both totally geodesic in M for a CR product.

10.0.2 Lemma

$h(JU, V) = h(U, JV)$.

Proof This has already been proven for almost Hermitian manifolds in general so we will not repeat the proof.

We further derive the following directly from the Gauss equation.

$$0 = 1+ < h(U,U), h(X,X) - < h(U,X), h(U,X) >, \tag{10.1}$$

$$0 = < h(U,JU), h(X,X) > - < h(U,X), h(JU,x) >, \tag{10.2}$$

$$0 = < h(U,JU), h(U,X) > - < h(U,U), h(JU,X) > . \tag{10.3}$$

where $U \in D$, $X \in D^{\perp}$, and both are unit vectors.

We shall prove the first of these as an example. Substituting the relevant vector fields into the Gauss equation we obtain:

$$\begin{aligned} < \tilde{R}(U,X)U,X > &= < R(X,X)Z,Z > + < h(X,X),h(U,U) > \quad (10.4) \\ & - < h(X,U),h(X,U) > \\ -1 &= 0+ < h(X,X),h(U,U) > \quad (10.5) \\ & - < h(U,X),h(U,X) > \\ 0 &= 1+ < h(X,X),h(U,U) > \quad (10.6) \\ & - < h(U,X),h(U,X) > . \end{aligned}$$

The other two follow in a similar fashion.

It is also useful to define a vector field $E \in \Gamma(D)$ relative to a choice of vector field $Z \in \Gamma(D^{\perp})$, by:

$$< h(E,E),h(Z,Z) >= \frac{Max}{< U,U >= 1, U \in D} < h(U,U),h(Z,Z) > .$$

We now prove the following :

10.0.3 Lemma

$h(E,Z) \neq 0$ and dim $M = 3$.

Proof Suppose that $h(E,Z) = 0$. Then from the Gauss identities (113 to 115) we have:

$$< h(E,E),h(E,Z) > +1 = 0.$$

However also note that we have:

$$< h(E,Z),h(E,Z) >= \frac{Max}{< U,U >= 1, U \in D} < h(U,U),h(Z,Z) > +1.$$

We deduce therefore that $< h(JE,Z),h(JE,Z) >= 0$, as this must be minimized and h is diagonalizable, and hence attains a minimum of zero.

Hence by the above Gauss identity (12):

$$0 = 1+ < h(JE,JE),h(Z,Z) > +0,$$
$$0 = - < h(E,E),h(Z,Z) > +1.$$

But this is a contradiciton and so we deduce that $h(E,Z) \neq 0$.

Now note that by a result due to Gray[13] there are no four dimensional holomorphic submanifolds of S^6. Hence we are forced to conclude that $dim(D) = 2$. Further by totally geodesic property of the leaves of $D^{\perp}$ we have

$$< h(JE,Z),JX >= 0 \forall X \in D^{\perp}$$

, and so $D^{\perp}$ has maximal dimension of 1. Hence $dimM = 3.//$

Further by counting dimensions we can infer that :

$$h(JE, Z) = aJh(E, Z),$$

for some $a \in \mathbf{R}$.

We now reach the main theorem of this chapter:

10.0.4 Theorem

There are no CR product submanifolds in S^6.

Proof We show that there is no a satisfying the requirement that

$$h(JE, Z) = aJh(E, Z)$$

, hence contradicting results for a product CR submanifold M, and so we deduce the non existence of such a submanifold. The full details are not of great interest in themselves and the reader is referred to Sekigawa[23] or Bejnacu's review text[3] for full details. The idea of the proof is to derive of number of contradictory statements concerning the hypothesised a. For example we have derived from the Gauss equation:

$$0 =< h(U, JU), h(U, X) > - < h(U, U), h(JU, X) > .$$

By substitution we then have:

$$\begin{aligned} 0 &= < h(E, JE), h(E, Z) > - < h(E, E), h(JE, Z) > & (10.7)\\ &= < JhE, E), h(E, Z) > - < h(E, E), aJh(E, Z) > & (10.8)\\ &= - < h(E, E), Jh(E, Z) > -a < h(E, E), Jh(E, Z) > & (10.9)\\ 0 &= (a + 1) < h(E, E), Jh(E, Z) > . & (10.10)\end{aligned}$$

By application of the CR submanifold theory already developed Sekigawa obtains a number of similar, contradictory expressions, hence proving the non-existence of a, and proving the non-existence of product CR submanifolds in S^6.

Chapter 11

Homogeneous Submanifolds of S^6

This chapter is largely a review of the paper by Hashimo, Mashimoto[15]. Clearly a submanifold generated by the image of some point under a 3-dimensional subgroup of G_2 will be a 3-dimensional homogeneous submanifold. In [15] each 3-dimensional subgroup of G_2 is considered (they are classified into four types), and tested for generating a manifold with CR structure. It shall be seen that the CR submanifold genrated may, or may not, depend on the initial point.

The following theorem gives a test for CR submanifolds:

11.0.5 Theorem

Let $e_1, ..., e_6$ be a basis for the standard complex space $\mathbf{C}^3$, with $e_4 = Je_1$, $e_5 = Je_2$, $e_6 = Je_3$, J the standard complex structure. Let $\omega_1, ...\omega_6$ be the dual basis to $e_1, ..., e_6$. Further define a lagrangean form ω by:

$$\omega = (\omega_1 + i\omega_4) \wedge (\omega_2 + i\omega_5) \wedge (\omega_3 + i\omega_6).$$

Given any three dimensional real subspace V of $\mathbf{C}^3$, then $dim_{\mathbf{R}}(V \cap J(V)) = 2$ if and only if $\omega(V) = 0$

Proof That this must be the case can be carried out by observing that if $\omega(V) \neq 0$ then V must be spanned by three vectors, one in each of $\{e_1, e_4\}$, $\{e_2, e_5\}$ and $\{e_3, e_6\}$. However by construction this will be a totally real subspace $dim_{\mathbf{R}}(V \cap J(V)) = 0$. Conversely if $\omega(V) = 0$ V must have empty interchapter with one of these subspaces, and so by dimension counting V will contain at least one of the preserved subspaces $\{e_1, e_4\}, \{e_2, e_5\}, \{e_3, e_6\}$ and so $dim_{\mathbf{R}}(V \cap J(V)) = 2$.//

And hence:

11.0.6 Corollary

For W a 6-dimensional (almost) complex manifold. A three dimensional submanifold M of W is a CR submanifold with one dimensional real distribution if and only if $\omega(T_x M) = 0$, $\forall x \in M$.

This follows immediately as the condition $dim_{\mathbf{R}}(T_x M \cap J(T_x M)) = 2$ is an assertion that some two dimensional subspace of the tangent space is preserved under the complex structure - i.e. the complex distribution D is non-empty at this point. If this is true $\forall x \in M$ then it follows that M is a CR submanifold of the required type.
Also we are interested in the existence of product and warped product examples. Hence it would be of interest if any of the subgroups contain a 2-dimensional subgroup which generates a holomoporhic submanifold at each point of the submanifold. The existence of such a subgroup would be a requirement for a CR (warped) product - this is not considered by Hashimo and Mashimoto, and it does not appear that the existence of such holomorphic submanifolds is trivial. There are four families of three dimensional simple subgroups of G_2. The following conclusions are reached. We describe each subgroup in terms of its subalgebra. We shall not give full details of the analysis, although in each case we apply the corollary above. We also take advantage of G_2 action to simplify calculations.

11.1 Orbits of Type I

This subalgebra has basis:

$$\begin{aligned} X_1 &= -A_{45} + A_{76}, \\ X_2 &= -A_{46} + A_{57}, \\ X_3 &= -A_{47} + A_{65}. \end{aligned}$$

(Where A_{ij} is the element of $so(7)$ which maps $e_j \to e_i$, $e_i \to -e_j$ and $e_k, k \neq i, j$ to 0) These orbits are small or great spheres, and recall from above that none of these are of CR type, except for a unique totally real submanifold based on the interchapter with e.g. $span < e_4, e_5, e_6, e_7 >$.

Orbits of Type II This subalgebra has basis:

$$\begin{aligned} X_1 &= -2A_{23} + A_{45} + A_{76}, \\ X_2 &= -2A_{31} + A_{46} + A_{57}, \\ X_3 &= -2A_{12} + A_{47} + A_{65}. \end{aligned}$$

This orbit generates a CR submanifold M, unique up to G_2 action. The CR subamnifold may be generated by taking for a base point the point x, with:

$$x_2^2 + x_3^2 = \frac{1}{9},$$

$$x_4^2 = \frac{8}{9}.$$

Every orbit through such a point is congruent to M by the transformation $exp(t(A_{23} - A_{76}))$, for some real t. Note that not all orbits of this subgroup have CR structure.

Orbits of Type III This subalgebra has basis:

$$X_1 = -2A_{21} - 2A_{65},$$
$$X_2 = -2A_{32} - 2A_{76},$$
$$X_3 = -2A_{31} - 2A_{75}.$$

It is shown that no CR submanifold is an orbit of an element of this subgroup.

Orbits of Type IV This subalgebra has basis:

$$X_1 = 4A_{32} + 2A_{54} + 6A_{76},$$
$$X_2 = \sqrt{6}(A_{37} + A_{26} - 2A_{15}) + \sqrt{10}(A_{42} - A_{35}),$$
$$X_3 = \sqrt{6}(A_{63} + A_{27} - 2A_{41}) + \sqrt{10}(A_{25} - A_{34}).$$

It is shown that the orbit of a point $\mathbf{x} \in S^6$is a CR submanifold if and only if the function f is identically zero. Where:

$$\begin{aligned} f(x) \quad = \quad & -5x_4^4 - 10x_4^2x_5^2 - 5x_5^4 + 42x_4^2x_7^2 + 42x_5^2x_7^2 \\ & -9x_7^4 - 24\sqrt{15}x_4^2x_5x_7 + 8\sqrt{15}x_5^3x_7. \end{aligned}$$

Further it is demonstrated that such submanifold form a two parameter family. This is demonstrated by considering the point x, with:

$$x_1 = \frac{1}{3},$$
$$x_7 = 2\frac{\sqrt{2}}{3},$$

(And other $x_i = 0$). For which $f = 0$, and notinng that the Jacobean of f at this point is regular with respect to x_1, x_7 on S^6, and hence we (locally) have a two parameter family of such submanifolds.

11.2 Extension to Higher Dimesnions

As used in [15] the form ω is an effective test for three dimensional CR-submanifolds where $dim_{\mathbf{R}} D = 2$. It is natural to wonder whether alternative forms might be of use in detecting other CR submanifolds. In the higher dimensional case:

$$\omega = (\omega_1 + i\omega_{n+1}) \wedge (\omega_2 + i\omega_{n+2}) \wedge ... \wedge (\omega_n + i\omega_{2n}).$$

Where $\omega_1, ...\omega_{2n}$ is a dual basis to $e_1, ..., e_{2n}$ an orthonormal basis of complex vector space $\mathbf{C}^n$, then for an n dimensional real vector space V, $dim(V \cap (JV)) \geq 2$ if and only if $\omega(V) = 0$. Hence any real submanifold M of n, where n is odd, s.t. $\omega(T_p M) = 0$, $\forall\ p \in M$ will be a non-trivial CR submanifold, although the dimensions of D, $D^\perp$ are not determined. Note the restrictions on both the ambient space and the submanifold that n is odd. The vanishing of this ω only indicates that part of the space behaves like a holomorphic subspace, in the case that n is even there is the possibility that M is purely holomorphic.

In the specific case of S^6 however we have the theorem due to Gray[13] that there are no 4-dimensional complex submanifolds of S^6. We still have the possibility that a submanifold may locally, or at a point, be completely holomorphic, so the vanishing of ω is not quite sufficient for 4-dimensional submanifolds.

The analysis of 4-dimensional subgroups of G_2 is therefore more problematic, although it would be immediately possible to demonstrate the non-existence of CR submanifolds using the method above.

Chapter 12

Some Warped Product CR-Submanifolds in S^6

12.1 Previous Examples of CR warped products

In [23] a warped product submanifold of S^6 is constructed of the form:

$$\begin{aligned} \Psi(y_2, y_4, y_6, e^{it}) &= (y_2 cost)e_2 - (y_2 sint)e_3 + (y_4 cos2t)e_4 + && (12.1)\\ & (y_4 sin2t)e_5 + (y_6 cost)e_6 - (y_6 sint)e_7. && (12.2)\\ & && (12.3) \end{aligned}$$

Where $y_2^2 + y_4^2 + y_6^2 = 1$, so that this is a mapping of $S^2 \times S^1$ into S^6. This is not an embedding as $\psi(y_2, y_4, y_6, e^{it}) = \psi(-y_2, y_4, -y_6, e^{i(t+\pi)})$. It is shown that this is indeed a CR submanifold with the distributions D and $D^{\perp}$ both integrable.

In [15] this is extended to a family of examples parameterised by $p_1, p_2, p_3 \in \mathbf{R}$, s.t. $p_1 + p_2 + p_3 = 0$ and $p_1 p_2 p_3 \neq 0$, given by:

$$\begin{aligned} \psi(x_1, x_2, x_3, t) &= exp(t(p_1 A_{51} + p_2 A_{62} + p_3 A_{73})(x_1 e_1 + x_2 e_2 + x_3 e_3)\\ &= x_1(cos(tp_1)e_1 sin(tp_1)e_5) + x_2(cos(tp_2)e_2 + sin(tp_2)e_6) +\\ &\quad x_3(cos(tp_3)e_3 + sin(tp_3)e_7). \end{aligned}$$

Where $x_1^2 + x_2^2 + x_3^2 = 1$, and $t \in \mathbf{R}$. We observe that this is a map $S^2 \times \mathbf{R} \to \mathbf{S^6}$. It is shown that the example in [23] is given by $p_1 = 2, p_2 = -1, p_3 = -1$ with some G_2 action. By the description of ψ it is clear that it is the orbit of a complex manifold S^2 under the one geodesic path in G_2 given by the g_2 algebra element $p_1 A_{51} + p_2 A_{62} + p_3 A_{73}$ (i.e. the path in G_2 formed by the exponential map acting on this algebra element). We shall demonstrate that this is in fact a specific example of a more general class of CR submanifolds.

It is also discussed in [15] as to under what circumstances this is a map of $S^2 \times \mathbf{R}$ and when it is a map of $S^2 \times S^1$. Certainly if the parameters p_1, p_2, p_3 are all co-rational (i.e. the ratios of any two are rational) then it is possible to find some value of $t \in \mathbf{R}$ s.t. tp_1, tp_2, tp_3 are all multiples of 2π and hence the map is indeed of $S^2 \times S^1$. However if the p_i are not co-rational, then no such t exists, and the mapping is instead that of $S^2 \times \mathbf{R}$. We may compare this situation with that of the irrational line on the torus.

12.2 General Warped Products in S^6

The examples given in [15], [23] have been shown to be the submanifolds generated by the orbit of a complex submanifold under some G_2 action. We naturally ask therefore how this may be generalised and what other warped product submanifolds may be formed from the orbit of some complex submanifold.

Let M^c be some complex submanifold of S^6, and let $\gamma : \mathbf{R} \times \mathbf{S^6} \to \mathbf{S^6}$ be some parameterised transformation on S^6, with $\gamma(0)$ the identity map. Let W be the submanifold consisting over the image of M^c under γ for all $\mathbf{R}$. If γ is to generate a warped product submanifold then we require:

a. γ preserves the metric on M^c

b. The distribution $\frac{d}{dr}\gamma(r)|_{r_0}M$ $\forall r_0 \in \mathbf{R}$ must be perpendicular to the tangent vector field $T(\gamma(r_0)M^c)$.

c. The submanifolds $\gamma(r)M^c$ must be complex submanifolds $\forall r \in \mathbf{R}$.

d. The variation is non-zero over all M^c

We will initially be concerned with constructing examples locally, and hence not immediately concerned with (d).

Consider for what variations this is true, i.e. for what perpendicular variations on a submanifold is the metric preserved:

12.2.1 Theorem

A submanifold $M \subset \tilde{M}$, $\tilde{M}$ a Riemannian manifold, is acted on by $\gamma : \mathbf{R} \times \tilde{\mathbf{M}} \to \tilde{\mathbf{M}}$, $\gamma(0)$ the identity map, and $\frac{d}{dr}\gamma(r)M^c$ is everywhere perpendicular to TM^c. Then γ preserves the metric on M^c (i.e. each $\gamma(r)M^c$ is isometric) if and only if $\frac{d}{dr}\gamma(r)|_{r_0}$ is everywhere perpendicular to the second fundamental form on $\gamma(r_0)M^c \forall r_0 \in \mathbf{R}$

proof It is sufficient to prove the result on a $\gamma(r_0)M^c$, and we can take this to be M^c itself without loss of generality. Consider a point $p \in M^c$, then there is an orthonormal basis e_i to the vector space T_pM^c. Let $\alpha_i(t)$ be paths in M^c through p s.t. $\alpha_i'(0) = e_i$. Assume that the variaton does peserve the metric. We analyse the situation locally:

$$
\begin{aligned}
< \frac{d}{dt}\gamma(r)\alpha_i, \frac{d}{dt}\gamma(r)\alpha_j > |_{t=0} &= \delta_{ij} \\
\frac{d}{dr} < \frac{d}{dt}\gamma(r)\alpha_i, \frac{d}{dt}\gamma(r)\alpha_j > |_{r=0,t=0} &= 0 \\
&= < \frac{d}{dr}\frac{d}{dt}\gamma(r)\alpha_i, \frac{d}{dt}\gamma(r)\alpha_j > + \\
&\quad < \frac{d}{dt}\gamma(r)\alpha_i, \frac{d}{dr}\frac{d}{dt}\gamma(r)\alpha_j > |_{r=0,t=0} \\
&= < \tilde{\nabla}_\xi e_i, e_j > + < e_i, \tilde{\nabla}_\xi e_j > \\
&= < A_\xi e_i, e_j > + < e_i, A_\xi e_j > \\
&= < \xi, h(e_i, e_j) > + < h(e_i, e_j), \xi > \\
&= 2 < \xi, h(e_i, e_j) > .
\end{aligned}
$$

Where we have written ξ for the vector field on M^c, $\frac{d}{dr}\gamma(r)M^c|_{r=0}$, and $\tilde{\nabla}$ for the metric connection on $\tilde{M}$. We have also used the fact that ξ is perpendicular to the tangent vector space on M, and the identity $< h(X,Y), \xi >=< A_\xi X, Y >$ where $A_\xi X$ is the tangential (w.r.t. M^c) component of $\tilde{\nabla}_\xi X$.

Hence the variation field must be perpendicular to the second fundamental form, at all points of the submanifold M. The proof of the converse is obtained by simply reversing the argument, and showing that $\frac{d}{dt} < e_i, e_j >= 0$ if it is assumed that $< \xi, h(e_i, e_j) >= 0, \forall i, j$ at all points of M^c.

12.3 Warped Products from Totally Geodesic S^2

We will intially examine the simplest example of warped product submanifolds in S^6, that of totally geodesic S^2 under the action of totally geodesic paths in G_2. Formally we take $M^c = S^6 \cap < e_1, e_2, e_3 >$ a complex submanifold. Further $\gamma(t) = exp(Xt)$ for some $X \in g_2$. With this choice γ necessarily preserves the metric on M (as $\gamma(t) \in SO(7)$) and by virtue of being in G_2 the complex structure on M^c is preserved. It is therefore only necessary to ensure that the variation field ξ is perpendicular to M^c.

Firstly we must define the relation between X and ξ. Let $p \in S^2$ then $p \rightarrow exp(Xt)p$ under the given transformation. Hence, $\xi(p) = \frac{d}{dt}exp(Xt)p|_{t=0} = Xp$. Let X be the most general element of g_2:

$$
X = \begin{pmatrix}
0 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & -a_6 \\
a_1 & 0 & -b_1 & -b_2 & -b_3 & -b_4 & -b_5 \\
a_2 & b_1 & 0 & -a_5 + b_3 & -a_6 - b_2 & a_3 - b_5 & a_4 + b_4 \\
a_3 & b_2 & a_5 - b_3 & 0 & -c_1 & -c_2 & -c_3 \\
a_4 & b_3 & a_6 + b_2 & c_1 & 0 & -a_1 - c_3 & -a_2 + c_2 \\
a_5 & b_4 & -a_3 + b_5 & c_2 & a_1 + c_3 & 0 & -b_1 - c_1 \\
a_6 & b_5 & -a_4 - b_4 & c_3 & a_2 - c_2 & b_1 + c_1 & 0
\end{pmatrix}.
$$

Let general point $p \in S^2$ be given by $p = x_1e_1 + x_2e_2 + x_3e_3$, where $x_1^2 + x_2^2 + x_3^2 = 1$. Now $\xi(p) = Xp$, and so:

$$\xi(p) = Xp = \begin{pmatrix} -a_1x_2 - a_2x_3 \\ a_1x_1 - b_1x_3 \\ a_2x_1 + b_1x_2 \\ a_3x_1 + b_2x_2 + (a_5 - b_3)x_3 \\ a_4x_1 + b_3x_2 + (a_6 + b_2)x_3 \\ a_5x_1 + b_4x_2 + (-a_3 + b_5)x_3 \\ a_6x_1 + b_5x_2 + (-a_4 - b_4)x_3 \end{pmatrix}.$$

Rather than examining than forcing $\xi(p)$ perpendicular to T_pS^2 for general $p \in S^2$, consider instead $p = e_1$, $x_1 = 1, x_2 = x_3 = 0$. At e_1, the tangent space is span$< e_2, e_3 >$, and hence we require:

$$0 =< \xi(e_1), e_2 >=< \xi(e_1), e_3 >$$

Writing these out:

$$\begin{aligned} < \xi(e_1), e_2 > &= a_1, \\ < \xi(e_1), e_3 > &= a_2. \end{aligned}$$

And so we deduce $a_1 = a_2 = 0$. We know also consider the point $p = e_2$, and observe that:

$$< \xi(e_2), e_3 >= b_1,$$

and so by similar arguments $b_1 = 0$. Hence $\xi(p)$ is reduced to:

$$\xi(p) = Xp = \begin{pmatrix} 0 \\ 0 \\ 0 \\ a_3x_1 + b_2x_2 + (a_5 - b_3)x_3 \\ a_4x_1 + b_3x_2 + (a_6 + b_2)x_3 \\ a_5x_1 + b_4x_2 + (-a_3 + b_5)x_3 \\ a_6x_1 + b_5x_2 + (-a_4 - b_4)x_3 \end{pmatrix}.$$

We observe that this $\xi(p)$ is everywhere perpendicular to the entire subspace *span* $< e_1, e_2, e_3 >$, and hence no further restrictions may be imposed. We only consider submanifolds as distinct when not related by some G_2 action, i.e. by a co-ordinate change. Recall that G_2 action allows free choice of some associative three plane, and a further perpendicular direction. We have already made the choice of the three plane e_1, e_2, e_3 in fixing a specific S^2, but there is still a free choice of an e_4 direction. For simplicity we will pick this such that $a_4 = a_5 = a_6 = 0$. Further we observe that by reparameterization of $\gamma(t)$ we art perfectly free to alter the length of ξ, hence we make pick $a_3 = 1$. Hence up to G_2 actions there is a four dimensional space of perpendicular variation fields $\xi(p)$ given by:

$$\xi(p) = Xp = \begin{pmatrix} 0 \\ 0 \\ 0 \\ x_1 + b_2x_2 + -b_3x_3 \\ b_3x_2 + b_2x_3 \\ b_4x_2 + (-1+b_5)x_3 \\ b_5x_2 - b_4x_3 \end{pmatrix}.$$

And derived from a variation $\gamma(t) = exp(Xt)$, for

$$X = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -b_2 & -b_3 & -b_4 & -b_5 \\ 0 & 0 & 0 & b_3 & -b_2 & 1-b_5 & b_4 \\ 1 & b_2 & -b_3 & 0 & -c_1 & -c_2 & -c_3 \\ 0 & b_3 & b_2 & c_1 & 0 & -c_3 & c_2 \\ 0 & b_4 & -1+b_5 & c_2 & c_3 & 0 & -c_1 \\ 0 & b_5 & b_4 & c_3 & -c_2 & c_1 & 0 \end{pmatrix}.$$

Hence we infer a seven dimensional family of CR submanifolds generated in this way.

12.4 Valid variation fields over the whole of S^2

Although we have demonstrated that there is a local family of CR submanifolds, generated by $S^2(e_1, e_2, e_3)$ under a seven dimensional famil of G_2 variations, it is not certain that this variation is everywhere non-zero.

Recall that we have the form for the varition at a point $p = (x_1, x_2, x_3) \in S^2$ given by:

$$\xi(p) = Xp = \begin{pmatrix} 0 \\ 0 \\ 0 \\ x_1 + b_2x_2 + -b_3x_3 \\ b_3x_2 + b_2x_3 \\ b_4x_2 + (-1+b_5)x_3 \\ b_5x_2 - b_4x_3 \end{pmatrix}.$$

Let us firstly rewrite into spherical polars, $x_1 = cos\phi$, $x_2 = \cos\theta\sin\phi$, $x_3 = \sin\theta\cos\phi$. The condition that $\xi(p) = 0$ is equivalent to solving the set of equations:

$$\cos\phi + (b_2\cos\theta - b_3\sin\theta)\sin\phi = 0, \tag{12.4}$$
$$(b_3\cos\theta + b_2\sin\theta)\sin\phi = 0, \tag{12.5}$$
$$(b_4\cos\theta + (b_5 - 1)\sin\theta)\sin\phi = 0, \tag{12.6}$$
$$(b_5\cos\theta - b_4\sin\theta)\sin\phi = 0. \tag{12.7}$$

We can rearrange these, assuming $\sin\phi \neq 0$, to the form:

$$b_2 = b_3 \tan\theta - \frac{\cot\phi}{\cos\theta}, \tag{12.8}$$
$$b_3 = -b_2 \tan\theta, \tag{12.9}$$
$$b_4 = (1 - b_5)\tan\theta, \tag{12.10}$$
$$b_5 = b_4 \tan\theta. \tag{12.11}$$

We may solve these equations as follows:

$$b_2 = -\cot\phi\cos\theta, \tag{12.12}$$
$$b_3 = \cot\phi\sin\theta, \tag{12.13}$$
$$b_4 = \sin\theta\cos\theta, \tag{12.14}$$
$$b_5 = \sin^2\theta. \tag{12.15}$$

These are certainly soluble for fixed θ, ϕ. The points $\sin\phi = 0$ correspond to the point $|x_1| = 1$, $x_2 = x_3 = 0$, at which point $\xi(p) \neq 0$ (in fact is in the e_4 direction). This should not be a surprise, as it is a consequence of some a non-vanishing vector at e_1 to be e_4.

We observe that for any other point $p \in S^2$ it is possible to select a G_2 action which is zero at this point. Also that opposite points on the sphere map to the same b_2, b_3, b_4, b_5. Further we observe that if we relax the condition that a_3 is identically 1, then we may normalize the vector a_3, b_2, b_3, b_4, b_5, and so observe that the map constructed appears to be a map from $\mathbf{R}P^1$ to $\mathbf{R}P^4$. However this map still has a singularity at $\sin\phi = 0$, where $a_3 = 0$, and b_2, b_3, b_4, b_5 are undefined.

If we remove our restriction that $a_4, a_5, a_6, a_7 = 0$, then we still see that at the point $x_1 = 1, x_2, x_3 = 0$, b_2, b_3, b_4, b_5 are undefined, and we actually have a mapping to a 4-plane in $\mathbf{R}^8$. Further in this situation there is no preferred point on $S^2(e_1, e_2, e_3)$, so we deduce that every point in $\mathbf{R}P^1$ maps to some 4-plane in $\mathbf{R}^8$. We deduce that the map of points of S^2 to their invalid points, is actually a map from $\mathbf{R}P^1$ to the space of 4-planes in $\mathbf{R}^8$.

Where we use G_2 to simplify matters, we implicitly prefer some directions, and reduce most of these 4-planes, to rays, or points, depending on whether results are normalized. We see that this reduction cannot be carried out smoothly over the whole sphere, hence the singularities occuring when we attempt to simplify. We deduce a mapping from 4-planes in $\mathbf{R}^8$ to $\mathbf{R}P^4$, except at the points corresponding to these singularities.

12.5 Suitablitiy of non-G_2 Variations

Here we demonstrate that the only varitions which produce warped product submanifolds on totally geodesic S^2 are locally generated by g_2 variations as above. We prove the following theorem:

12.5.1 Theorem

Let M^c be some totally geodesic holomorphic S^2 in S^6. Let $\gamma(t)$ be some path in $GL(7)$, such that $\gamma(t)M^c$ is holomorphic and isometric to M^c. Then $\frac{d}{dt}\gamma(t)M^c|_{t_0}$ is equal to $\frac{d}{dt}\gamma'(t)M^c{}_{t_0}$ for some γ' a path in G_2 dependant on t_0, for each t_0 over which γ is defined.

Proof This is a purely local result, so it is enough to consider a fixed $t_0 = 0$. Further it is enough to consider γ' to be geodesic, i.e. the exponential map of some element of g_2. We also require the fact that the only isometric manifolds to totally geodesic S^2 in S^6, are other totally geodesic S^2. Hence we see that γ will necessarily transform assosciative 3-planes to assosciative 3-planes, and given any such motion we can find a path in G_2 which moves the relevant 3-planes in the required fashion (this is a basic property of G_2, and has been implicitly used when we move assosciative three planes to e_1, e_2, e_3). The only difficulty would be some non-SO(3) varition within the 3-plane, but this would not be metric preserving on M^c//

Note that there is no restriction on the action of γ off the image of M^c, but this is of no consequence for the generated submanifold M. Also note that although locally γ may be approximated by a totally geodesic γ', it is not necessarily globally totally geodesic, and we shall consider this case.

Note further that we are in the restricted case where M^c is some S^2. In order to extend to more complicated M^c it would be necessary to prove that all isometric holomorphic submanifolds to M^c are related by some G_2 variation locally. The converse, that holomrophic submanifolds under some G_2 variation are isometric and holomorphic, naturally holds regardless, and so defines at least some subset of warped product submanifolds.

12.6 Second Fundamental Form of Warped Product Examples

We know consider the properties of this family of submanifolds. We construct a CR submanifold M as the orbit of S^2 under a suitable G_2 geodesic variation as above. We calculate the second fundamental form on the base manifolds S^2, in terms of spherical polar coordinates θ, ϕ, on S^2, where $x_1 = \cos\theta\cos\phi, x_2 = \sin\theta\cos\phi, x_3 = \sin\phi$. First of all we note that the second fundamental form of great spheres in S^6 is identically zero (although along radial directions in $\mathbf{R}^7$). Hence the second fundamental form of M between vectors in some S^2 is identically zero. i.e.

$$h(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\phi}) = 0,$$

$$h(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta}) = 0,$$

$$h(\frac{\partial}{\partial \phi}, \frac{\partial}{\partial \phi}) = 0.$$

Remembering that h is a symmetric function.

Recall that the second fundamental form $h(X,Y)$ is given by the perpendicular part of a covariant derivation $\tilde{\nabla}_X Y$. Hence we first need the tangent vectors, $\frac{\partial}{\partial \theta}$ and $\frac{\partial}{\partial \phi}$. xi is already known explicitly.

$$\begin{aligned}
\frac{\partial}{\partial \theta} &= \frac{\partial x_i}{\partial \theta}\frac{\partial}{\partial x_i} \\
&= (-\sin\theta\cos\phi)\frac{\partial}{\partial x_1} + (\cos\theta\cos\phi)\frac{\partial}{\partial x_2}.
\end{aligned}$$

And similarly:

$$\begin{aligned}
\frac{\partial}{\partial \phi} &= \frac{\partial x_i}{\partial \phi}\frac{\partial}{\partial x_i} \\
&= (-\cos\theta\sin\phi)\frac{\partial}{\partial x_1} - \\
&\quad (\sin\theta\sin\phi)\frac{\partial}{\partial x_2} + \\
&\quad (\cos\phi)\frac{\partial}{\partial x_3}.
\end{aligned}$$

Now we calculate the relvant covariant derivative $\tilde{\nabla}_\xi \xi$ at some point :

$$\begin{aligned}
\tilde{\nabla}_\xi \xi &= \frac{d}{dt}(\xi(exp(t\xi(p)))|_{t=0} \\
&= \frac{d}{dt}(Xexp(t\xi(p)))|_{t=0} \\
&= (X\xi(p)exp(t\xi(p)))|_{t=0} \\
&= (X\xi(p)) \\
&= X^2 p.
\end{aligned}$$

Where X^2 is calculated by matrix multiplication.

We calculate that:

$$X^2 p = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -c_1k_1 - c_2k_3 - c_3k_4 \\ c_1k_1 - c_3k_3 + c_2k_4 \\ c_2k_1 + c_3k_2 - c_1k_4 \\ c_3k_1 - c_2k_2 + c_1k_3 \end{pmatrix}.$$

Where k_i are expressed by:

$$k_1 = x_1 + b_2 x_2 - b_3 x_3, \tag{12.16}$$
$$k_2 = b_3 x_2 + b_2 x_3, \tag{12.17}$$
$$k_3 = b_4 x_2 + (b_5 - 1) x_3, \tag{12.18}$$
$$k_4 = b_5 x_2 - b_4 x_3. \tag{12.19}$$

We calculate the $h(\xi, \xi)$ by removing parts parallel to ξ, however we need not do this to observe that $h(\xi, \xi)$ will be identically zero over the whole of S^2 if and only if $c_1 = c_2 = c_3 = 0$. Recall that the second fundamental form on S^2 is already identically zero, and so we deduce that the CR subamnifold M is minimal if and only if $c_i = 0$. Further note that c_i do not occur in the description of that tangent field on S^2. We reach the following conclusions about the CR submanifolds constructed:

1. The variation fields on S^2 form a four dimensional connected family (taking into account some space of invalid variations).

2. Corresponding to each variation field there is a three dimensional family of CR submanifolds (due to the choice of c_i), of each exactly one is minimal.

We now consider the diagonal part of the second fundamental form by considering $\xi(p)$ as a function of θ, ϕ. It is simple to calculate that:

$$\frac{\partial \xi}{\partial \theta} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} (-\sin\theta\cos\phi) + \begin{pmatrix} 0 \\ 0 \\ 0 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{pmatrix} (\cos\theta\cos\phi),$$

and,

$$\frac{\partial \xi}{\partial \phi} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} (-\cos\theta\sin\phi) + \begin{pmatrix} 0 \\ 0 \\ 0 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{pmatrix} (-\sin\theta\sin\phi) + \begin{pmatrix} 0 \\ 0 \\ 0 \\ -b_3 \\ b_2 \\ -1 + b_5 \\ -b_4 \end{pmatrix} (\cos\phi).$$

The second fundamental form is calculated by removing the parts of these tangential to the manifold, i.e. by removing the parts along $\frac{\partial}{\partial \theta}$, $\frac{\partial}{\partial \phi}$ and $\frac{\partial}{\partial \xi}$. Consider these quanitities evaluated at $\sin\phi = 0$, $\cos\phi = 1$, i.e. around some great circle parameterised by θ. It is immediately seen that $\frac{\partial \xi}{\partial \phi}$ is constant along this circle, dependant on the b_i chosen. Consider further the form derived for $\frac{\partial \xi}{\partial \theta}$ and we see that at $\cos\theta = 1$ the value is constant, independant of the chosen

variation, although away from this point there is a dependancy on b_i. We see therefore that the non-diagonal entries of the second fundamental form is non-constant, and of non-constant length.
In general we see therefore that the second fundamental form is non-constant, and argue from this that the scalar curvature of the submanifold is non constant (from the Gauss equation relating the second fundamental form to the curvature tensor). Whether the b_i, c_i might be chosen so as to generate a submanifold with constant curvature is left as an open question.

12.7 Totally real parts in S^1 and $\mathbf{R}$

As observed the examples in [15] and [23] were shown to be maps of $S^2 \times S^1$ or $S^2 \times \mathbf{R}$ depending on whether certain coefficients are corational. Certainly in our more general case if all components of the g_2 variation element, X, are corational then the resultant submanifold will indeed be the image of $S^2 \times S^1$. Simply observe that under the exponential map, although the full form is more complicated than that of [15], the result will consist of trigonometric functions with co-efficients multiples and sums of the components of the given $X \in g_2$ - and hence all corational.

The converse does not immediately follow - it is only certainly necessary that coefficients of $exp(Xt)$ are corational, or equivalently that there exists some t_0 s.t. $exp(Xt_0)$ is the identity. The form of $exp(Xt)$ involving trigonometric functions will not involve all combinations of elements of X and so it is not absoltutely certain that every pair elementt of X be corational. Certainly examples of $S^2 \times \mathbf{R}$ exist (e.g. the example in [15]), and are not difficult to construct.

We note that these comments only apply to the case where the holomorphic submanifold M^c is acted on by a geodesic path in G_2. For a more arbitrary path γ, then the distinction between the two cases is only dependant on the existence of a t_0 s.t. $\gamma(t) = \gamma(t + t_0)$, for the mapping to be that of $S^2 \times S^1$. Further the case of $S^2 \times \mathbf{R}$ is indicated by γ taking distinct values for all values of $t \in \mathbf{R}$. We also note that these are not the only two cases possible.

12.8 Non-geodesic Paths in G_2

All examples explicitly constructed so far have been through totally geodesic paths, M^c under the action $exp(Xt)$ for $t \in \mathbf{R}$ and $X \in g_2$, with X subject to the given restrictions. However the analysis has all been performed locally, and it would certainly be possible to envisage a path $\gamma(t)$ in G_2 which is not of this type. This would be CR as long as $\frac{d}{dt}\gamma(t)M^c|_{t_0}$, an element of g_2, satisfies the required restrictions w.r.t. the submanifold $\gamma(t_0)M^c$ (i.e. that the variation is perpendicular to the subamnifold and its tangent space at each point). Certainly we have demonstrated there is a 4-dimensional choice of valid variation fields on each leaf, so the choice of a valid such γ is not too heavily constrained. We deduce that it is possible to construct warped product CR submanifolds which are locally identical, but not globally.

As we are more concerned with the submanifold M, rather than the exact details of construction we have ignored any consideration of the length of the variation field, equivalent to the velocity of the path in G_2. Certainly the generated submanifold is independant velocity of the path, except in the special case where at either or both ends the velocity of the path tends to zero faster than a critical value. As an example with $X \in g_2$, generating a valid varition on M^c, consider the following two cases:

$$M = exp(Xt)M^c,$$

$$M = exp(X \tan^{-1} t)M^c.$$

The first case is that considered previously. In the second case we note that $\tan^{-1}$ is restricted to the interval $(-\pi/2, +\pi/2)$, and hence the mapping is in reality the image of $M^c \times (0,1)$, i.e. some open interval - a distinct CR submanifold to those considered before. It can be seen that any smooth mapping could be appplied to the t variable, and so produce CR submanifolds which are warped products of M^c and open intervals or semi-open intervals. The local properties of such manifolds remain unchanged.

We note that any warped product CR submanifold is locally equivalent to some warped product CR submanifold where the real part is generated by a geodesic variation, although not uniquely. Further in the case where the holomorphic part is S^2 there is a correspondane to a unique minimal submanifold with geodesic real part. The existence of minimal CR submanifolds which are warped products of $S^2 \times \mathbf{R}$ with non-geodesic real parts are certainly possible.

12.9 Four dimensional Warped Product Examples

We now consider whether it is possible to create higher dimension CR submanifolds using the methods so far employed. Let us consider therefore a CR submanifold M which is the orbit of some complex submanifold M^c under the map $exp(Xt)$, where $X \in g_2$ subject to the given restrictions due to M^c. Suppose that there were a further $Y \in g_2$, would M^c under the double variation $exp(Xt)exp(Ys)$ be a 4-dimensional CR subamnifold with 2-dimensional real part?

Certainly the generated submanifold is a warped product submanifold of the type $M^c \times \mathbf{R}^2$. We require that both X and Y are valid variations for every value of s, t. We consider the case where M^c is a totally geodesic S^2 as above, and so we have a specific form that X, Y may take at $s = 0, t = 0$. We now observe that X, Y must be both of the form

$$\begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -b_2 & -b_3 & -b_4 & -b_5 \\ 0 & 0 & 0 & b_3 & -b_2 & 1-b_5 & b_4 \\ 1 & b_2 & -b_3 & 0 & -c_1 & -c_2 & -c_3 \\ 0 & b_3 & b_2 & c_1 & 0 & -c_3 & c_2 \\ 0 & b_4 & -1+b_5 & c_2 & c_3 & 0 & -c_1 \\ 0 & b_5 & b_4 & c_3 & -c_2 & c_1 & 0 \end{pmatrix}.$$

And perpendicular to each other. It is possible to demonstrate that these two conditions are incompatible, and so we cannot generate a 4-dimensional CR submanifold of this form. Further as we have shown that any warped product CR submanifold on a holomorphic S^2 must have its real variation generated locally by g_2 elements. Hence we deduce that there are no 4-dimensional CR warped product submanifolds of S^6 with holomorphic part a totally geodesic S^2.

12.10 Further Ideas

The study of warped product submanifolds has been shown to be a way of easily generating CR submanifolds in S^6, and there are several ways in which this work might be extended.

12.10.1 Further Analysis of Given Examples

The analysis of the examples given is far from exhaustive, and there are many extensions which immediately suggest themselves from comparison with other work. For example it would be possible to consider how the constructed examples have foliate, or mixed foliate structure. Further we may consider whether such submanifolds are linearly full in S^6 (not contained in some interchapter of a linear subspace with S^6). We would like to consider whether any of the examples given are holomorphic, and consider other submanifold properties.

12.10.2 More General Base Manifolds

We have considered only the restricted case of CR submanifolds generated from the orbit of a totally geodesic holomorphic submanifold. Ideally we would like to have a general theory for orbits of an arbitrary holomorphic submanifold. Although a start could be made by considering a specific holomorphic manifold, it would be more useful to consider a general such, through a local analysis. It has been suggested that such an analysis might be facilitated by the consideration of harmonic maps, similar to the work in Bolton, Vrancken and Woodward[8] or Bolton and Woodward[9], where harmonic maps are succesfully applied to the study of holomorphic submanifolds in S^6.

Part IX

Conclusions

We have seen that the study of the existence of CR product submanifolds has reached a sufficient stage as to make general statements about the existence of CR products in various spaces. Further the construction of such spaces in $\mathbf{C}^n and \mathbf{C}P^n$ is well understood.

The study of CR warped products is less well understood, although we have demonstrated existence, explicitly constructed a family of such submanifolds in S^6. As far as we know there has been no study of CR warped products in general Kaehler manifolds, and this may provide a useful area for further study. Whether the construction methods employed in S^6 might be extended to other manifolds is an open question.

The study of CR submanifolds with non-integrable real and complex leaves is relatively untouched. We have the general morse theory ideas due to Lei, Wolfson[18]. The homogeneous examples in S^6 [15] do not rely on integrability of the distributions, although integrability is not explicitly considered. Other papers have considered general properties, such as [16] due to Hashimoto, Mashimo and Sekigawa, where some topological restriction are placed on general 4-dimensional CR submanifolds. There is not however a comprehensive classification of such submanifolds, and examples are few compared those with integrable leaves.

We note that there has not been sufficient space for all of the related results developed in the past thirty years, however we hope that we have given a flavour of the results which are possible in the field, and an indication of some further results. We are confident that this is the first time that these particular results have been collected and compared in a single paper.

This work has hopefully demonstrated that the existence of a CR structure has interesting consequences for its relation with the ambient manifolds. The fact that CR submanifolds are readily understandable once holomorphic and real manifolds have been considered is in its favour. Considering the large body of work relating to these submanifolds, and the success with which results have so far been obtained for CR sumanifolds, it seems highly likely that further study will be rewarded. We have indicated in the text where we have seen immediate opportunities for research, particularly in the field of warped product submanifolds, but almost any text studying holomorphic submanifolds will suggest possible extensions.

Part X

References

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